Let \( q : X \to Y \) and \( r : Y \to Z \) be covering maps; let \( p = r \circ q \). Show if \( r^{-1}(z) \) is finite for each \( z \in Z \), then \( p \) is a covering map.

SOLUTION. Given \( z \in Z \), we show that there is an open neighborhood \( A \) of \( z \) which is even with respect to \( r \circ q \). Let \( U \) be an even neighborhood of \( z \) with respect to \( r \). Then
\[
 r^{-1}(U) = \prod_{j \in J} V_j
\]
with open \( V_j \subset Y \) such that the restriction of \( r \) to each \( V_j \) is a homeomorphism of \( V_j \) onto \( U \). For each \( j \in J \) we denote by \( y_j = r^{-1}(z) \cap V_j \) the point that corresponds to \( z \). Fix an open neighborhood \( W_j \subset V_j \) of \( y_j \) even with respect to \( q \). Thus,
\[
 q^{-1}(W_j) = \coprod_{i \in I_j} O_{ij}
\]
where \( O_{ij} \) are open in \( X \) and the restriction \( q_{|O_{ij}} : O_{ij} \to W_j \) is a homeomorphism for all \( i \) and \( j \). Note that each set \( r(W_j) \) is an open neighborhood of \( z \). We define
\[
 A = \bigcap_{j \in J} r(W_j).
\]
By the problem assumption we have that \( J \) is finite. Hence \( A \) is open. Denote the open sets \( r^{-1}(A) \cap V_j \) by \( A_j \) and the open sets \( q^{-1}(A_j) \cap O_{ij} \) by \( B_{ij} \). We show that \( A \) is even with respect to \( r \circ q \). Indeed,
\[
 (r \circ q)^{-1}(A) = q^{-1}(r^{-1}(A)) = q^{-1} \left( \prod_{j \in J} A_j \right) = \prod_{j \in J} q^{-1}(A_j) = \prod_{j \in J} \coprod_{i \in I_j} B_{ij}
\]
and the restriction \( (r \circ q)|_{B_{ij}} : B_{ij} \to A \) is a homeomorphism as the composition of two homeomorphisms \( q_{|B_{ij}} : B_{ij} \to A_j \) and \( r_{|A_j} : A_j \to A \).