1. Show that \( \mathbb{R}P^3 \) is not homotopy equivalent to \( \mathbb{R}P^2 \vee S^3 \).

SOLUTION: The reduced cohomology ring \( \tilde{H}^*(\mathbb{R}P^2 \vee S^3; \mathbb{Z}_2) \) = \( \tilde{H}^*(\mathbb{R}P^2; \mathbb{Z}_2) \oplus \tilde{H}^*(S^3; \mathbb{Z}_2) = \mathbb{Z}_2[\alpha]/(\alpha^3) \{1\} \oplus \mathbb{Z}_2[\beta] \{1\} \) where \( x^3 = 0 \) for every element \( x \). On the other hand, in the ring \( \tilde{H}^*(\mathbb{R}P^2; \mathbb{Z}_2) = \mathbb{Z}_2[\gamma]/(\gamma^4) \{1\} \) we have \( \gamma^3 \neq 0 \).

2. Show that for any closed connected orientable \( n \)-manifold \( M \) there is a degree one map \( M \to S^n \).

SOLUTION: Let \( B \subset \subset \mathbb{R}^n \subset M \) be a closed ball and Let \( D \subset \text{Int}B \) be a smaller closed ball. Then by the Homotopy Invariance the inclusion homomorphism \( H_n(M, M \setminus \text{Int}B) \to H_n(M, M \setminus D) \) is an isomorphism. By Theorem proven in class, the inclusion homomorphism \( H_n(M) \to H_n(M, M \setminus D) \) is an isomorphism. Then the inclusion homomorphism \( H_n(M) \to H_n(M, M \setminus \text{Int}B) \) is an isomorphism. Then we claim that the quotient map \( q : M \to M/(M \setminus \text{Int}B) \cong B/\partial B \cong S^n \) has degree one. This follows from the commutative diagram generated by exact sequences of pairs \( (M, M \setminus \text{Int}B), (S^n, pt) \) and the map \( q \)

\[
\begin{array}{ccc}
H_n(M) & \xrightarrow{\cong} & H_n(M, M \setminus \text{Int}B) \\
q_* \downarrow & & q_* \downarrow \cong \\
H_n(S^n) & \xrightarrow{\cong} & H_n(S^n, pt)
\end{array}
\]

Thus, \( q_* : H_n(M) \to H_n(S^n) \) is an isomorphism and, hence, \( \text{deg}(q) = \pm 1 \).