Topology 2 QUIZ 3 SOLUTIONS:

1. Show that for any closed connected orientablle n-manifold M there is a map $f : M \to S^n$ of degree 1, i.e., satisfying the equality $f_*([M]) = [S^n]$ for the fundamental classes.

SOLUTION. Let $B \subset M$ be a closed subset homeomorphic to the n-ball. Let $q : M \to M/(M \setminus Int(B)) = B/\partial B = S^n$ be the quotient map. We show that $deg(q) = 1$.

Since M is orientable, the inclusion homomorphism $i' : H_n(M) \to H_n(M|B)$ is an isomorphism. The inclusion homomorphism $i : H_n(M) \to H_n(M, M\setminus Int(B))$ is the composition $i = i'' \circ i'$, where $i'' : H_n(M|B) = H_n(M, M \setminus B) \to H_n(M, M \setminus Int(B))$ is an isomorphism being induced by a homotopy equivalence of the pairs $(M, M\setminus B) \to (M, M\setminus Int(B))$.

In view of excision the inclusion homorphism $j : H_n(B, \partial B) \to H_n(M, M \setminus Int(B))$ is an isomorphism

The map q brings the commutative diagram

$$\begin{array}{ccccc} H_n(M) & \xrightarrow[\cong]{i} & H_n(M, M \setminus Int(B)) & \xleftarrow[\cong]{j} & H_n(B, \partial B) \\ {\scriptstyle q_*}\downarrow & & {\scriptstyle q'_*}\downarrow & & {\scriptstyle q''_*}\downarrow\cong \\ H_n(S^n) & \xrightarrow{=} & H_n(S^n) & \xleftarrow{=} & S^n \end{array}$$

which implies that q_* is an isomorphism that takes $[M]$ to $[S^n]$.

2. Show that if a closed orientable manifold M of dimension $2k$ has $H_{k-1}(M; \mathbb{Z})$ torsion free, then $H_k(M; \mathbb{Z})$ is also torsion free.

SOLUTION. We use notation TA for the torsion subgroup of an abelian group A.

By the Poincare Duality $TH_{k-1}(M) = TH^{k+1}M$. By the UCT $TH^{k+1}M = TH_k(M)$. Thus, if $TH_{k-1}(M) = 0$, then $TH_k(M) = 0$.