1. Let $A$ be a compact set in a metric space $X$. Let $U$ be an open neighborhood of $A$.
(a) Show that there is $\epsilon > 0$ such that the $\epsilon$-neighborhood of $A$ is contained in $U$.
(b) Show that (a) need not hold true if $A$ is closed but not compact.

SOLUTION:
(a) The distance function $f(x) = d(x, X \setminus U)$ is a continuous function $f : A \to \mathbb{R}$. By the Extreme Value theorem there is $a \in A$ such that $f(a) = \min f(x)$. Since $U$ is open, there is a ball $B(a, \epsilon)$ containing in $U$. Show that $U_\epsilon(A) \subset U$. Since for all $y \in U_\epsilon(A)$ there is $x \in A$ such that $d(y, x) < \epsilon \leq d(x, X \setminus U)$, it follows that $y \in U$.
(b) Let $X = \mathbb{R}^2$, $A = \{0\} \times \mathbb{R}$, and $U = \{(x, y) \mid |y| < 1/|x|\}$.

2. Show that if $Y$ is compact then the projection $\pi_1 : X \times Y \to X$ is a closed map.

SOLUTION: Let $A \subset X \times Y$ be a closed subset. To show that $\pi_1(A) \subset X$ is closed we show that its complement is open. Let $x \in X \setminus \pi_1(A)$. Then $x \times Y \subset (X \times Y) \setminus A$. By the Tube Lemma there is an open neighborhood $U$ of $x$ such that $U \times Y \subset (X \times Y) \setminus A$. Therefore $U \subset X \setminus \pi_1(A)$. Since $x$ is arbitrary, it follows that $(X \times Y) \setminus A$ is compact.