MAP 2302  Spring-2019  Quiz 5  SOLUTION:

Solve the initial value problem

\[ y''' + 2y'' - 9y' - 18y = -18x^2 - 18x + 22; \quad y(0) = -2, \quad y'(0) = -8, \quad y''(0) = -12. \]

SOLUTION: The homogeneous equation is \((D^3 + 2D^2 - 9D - 18)[y] = 0\). It can be factorized as \((D + 2)(D - 3)(D + 3)[y] = 0\). Hence a general solution to the homogeneous equation is

\[ y_h = C_1 e^{-2x} + C_2 e^{3x} + C_3 e^{-3x}. \]

Note that \(D^3\) annihilates the right-hand side. We apply \(D^3\) to the equation to obtain

\[ D^3(D + 2)(D - 3)(D + 3)[y] = 0. \]

A general solution to it is

\[ y = C_1 e^{-2x} + C_2 e^{3x} + C_3 e^{-3x} + c_4 x + c_5 x^2 = y_h + y_p. \]

Thus, \(y_p = c_4 + c_5 x + c_6 x^2\). To find \(c_4, c_5,\) and \(c_6\) we compute the derivatives

\[ y'_p = c_5 + 2c_6 x, \quad y''_p = 2c_6, \quad \text{and} \quad y'''_p = 0 \]

and plug them to the equation to obtain

\[ 0 + 4c_6 - 9c_5 - 18c_6 x - 18c_4 - 18c_5 x - 18c_6 x^2 = -18x^2 - 18x + 22. \]

From that we obtain: 

\[ -18c_6 = -16 \quad \text{and, hence,} \quad c_6 = 1; \]

\[ -18c_6 - 18c_5 = -18 \quad \text{which is equivalent to} \quad -18c_5 = 0 \quad \text{and, hence,} \quad c_5 = 0; \]

\[ 4c_6 - 9c_5 - 18c_4 = 22 \quad \text{which is equivalent to} \quad 4 - 18c_4 = 22 \quad \text{and, hence} \quad c_4 = -1. \]

Thus, a general solution to the equation is

\[ y = C_1 e^{-2x} + C_2 e^{3x} + C_3 e^{-3x} + x^2 - 1. \]

We compute the derivatives
\[ y' = -2C_1 e^{-2x} + 3C_2 e^{3x} - 3C_3 e^{-3x} + 2x, \]
\[ y'' = 4C_1 e^{-2x} + 9C_2 e^{3x} + 9C_3 e^{-3x} + 2 \]

and plug the initial conditions to obtain the system:

\[ C_1 + C_2 + C_3 - 1 = -2; \]
\[ -2C_1 + 3C_2 - 3C_3 = -8; \]
\[ 4C_1 + 9C_2 + 9C_3 + 2 = -12. \]

We multiply the first equation by 9 and subtract from the third to obtain

\[ C_1 = 1. \]

Plug it to the first two equations to obtain the system

\[ C_2 + C_3 = -2; \]
\[ 3C_2 - 3C_3 = -6. \]

Therefore, \( C_3 = 0 \) and \( C_2 = -2. \)

The solution to the initial value problem is

\[ y = e^{-2x} - 2e^{3x} + x^2 - 1. \]