On the dimension of composition posets

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**Dimension**

- Given a countable poset $P$, the *dimension* of $P$ is the least positive integer $k$ so that $P$ embeds in $\mathbb{R}^k$. 
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- Dimension is monotone, meaning the dimension of a poset is at least that of each of its subposets.
- The poset of words of a length \( k \) over the positive integers, \( \mathbb{P}^k \), has dimension \( k \).
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- Given two compositions, we say $a(1)a(2)\ldots a(k) \leq b(1)b(2)\ldots b(n)$ if there are indices $1 \leq i_1 < i_2 < \cdots < i_k \leq n$ such that $a(j) \leq b(i_j)$ for each $j$. 

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![Skyline Diagrams](https://via.placeholder.com/150)
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- Given a possibly infinite composition \(u\), the age of \(u\) is the set of (finite) compositions which embed into it.

- For example, we say \(114221 \in \text{Age}(1^{\omega} \omega 2131^{\omega})\) as exhibited by the embedding below.
Lower Bounds on Dimension
LOWER BOUNDS ON DIMENSION

- Crown of dimension $n$:
Examples

- $\text{Age}(\omega)$ has dimension 1.
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- $\text{Age}(\omega)$ has dimension 1.
- $\text{Age}(\omega \omega)$ has dimension 3.
  - $\text{Age}(\omega \omega)$ has dimension at least 3:

```
13
21

31
12

22
3
```
Examples

▶ Age($\omega\omega\omega$)
Examples

- Age(ωωω)
  - Infinite dimensional!
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  - $n \geq 5$, crown of dimension $n - 3$:

\[
\begin{align*}
1n(n - 3) & \quad 2n(n - 4) & \quad 3n(n - 5) & \quad \cdots & \quad (n - 3)n1 \\
2(n - 2) & \quad 3(n - 3) & \quad 4(n - 4) & \quad \cdots & \quad (n - 2)2
\end{align*}
\]
Theorem

A downset of compositions in the generalized subword order is finite dimensional if and only if it does not contain $\omega^\omega$, $\omega^\omega\omega\omega$, $\omega\omega\omega^\omega\omega$, or $\omega\omega\omega\omega\omega$. 
THEOREM

Theorem
A downset of compositions in the generalized subword order is finite dimensional if and only if it does not contain \( \text{Age}(\omega \omega \omega) \), \( \text{Age}(1^\omega 2^\omega 1^\omega) \), \( \text{Age}(\omega 1^\omega \omega 1^\omega) \), or \( \text{Age}(1^\omega \omega 1^\omega \omega) \).
Age\((1^\omega 21^\omega 21^\omega)\)

- \(n \geq 5\), crown of dimension \(n - 3\):
Age(\(\omega_1^\omega \omega_1^\omega\))

- \(n \geq 3\), crown of dimension \(n - 1\):

\[
\begin{align*}
&11^0 n1^{n-1} \\
&21^n \\
&21^1 n1^{n-2} \\
&31^n \\
&31^2 n1^{n-3} \\
&41^{n-2} \\
&\cdots \\
&(n - 1)1^{n-2} n1^1 \\
&n1^2
\end{align*}
\]
### A Brief Sketch

Higman's Lemma and a theorem of Fra¨ıss´e show that every downset of compositions is the finite union of ages.

The union of two finite dimensional downsets is finite dimensional.

Maximal finite dimensional ages:
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- Maximal finite dimensional ages:
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**Example.** Let P be the poset of integer partitions, whose order is simply the one in Young’s lattice, namely containment of Ferrers diagrams. We establish the following result.
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Thank you.