On the dimension of composition posets

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- The poset of words of a length k over the positive integers, $\mathbb{P}^k$, has dimension k.

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- Given two compositions, we say $a(1)a(2)\dots a(k) \leqslant b(1)b(2)\dots b(n)$ if there are indices $1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant n$ such that $a(j) \leqslant b(i_j)$ for each j.

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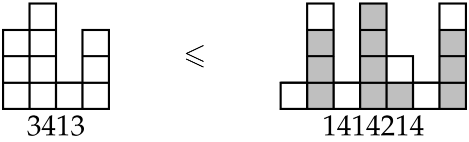
- This order can be visualized using *skyline diagrams*!



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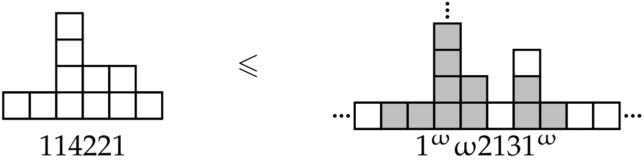
- For example, we say $114221 \in \text{Age}(1^\omega \omega 2131^\omega)$ as exhibited by the embedding below.



Lower Bounds on Dimension

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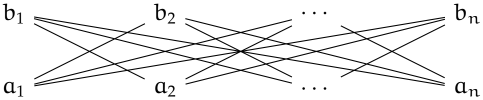
- Crown of dimension n:



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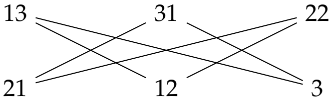
- Age(ω) has dimension 1.
- Age($\omega\omega$) has dimension 3.
  - Age($\omega\omega$) has dimension at least 3:



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- $\mathrm{Age}(\omega\omega\omega)$

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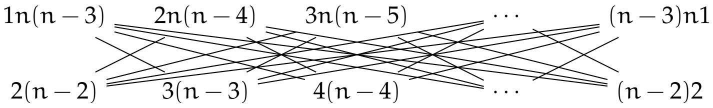

Theorem

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A downset of compositions in the generalized subword order is finite dimensional if and only if it does not contain $\mathrm{Age}(\omega\omega\omega)$, $\mathrm{Age}(1^\omega 21^\omega 21^\omega)$, $\mathrm{Age}(\omega 1^\omega \omega 1^\omega)$, *or* $\mathrm{Age}(1^\omega \omega 1^\omega \omega)$.

$\mathrm{Age}(1^\omega 2 1^\omega 2 1^\omega)$

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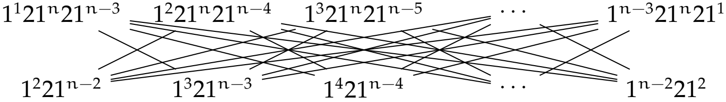

Age($\omega 1^\omega \omega 1^\omega$)

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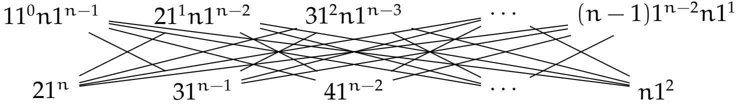

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- The union of two finite dimensional downsets is finite dimensional.
- Maximal finite dimensional ages:
  - $\mathrm{Age}(a\omega b 1^{\omega} c 1^{\omega} d \omega e)$
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Thank you.