16. Use differentials to approximate $\sqrt{16.328}$.

Let $f(x) = \sqrt{x}$. We will use a tangent line to $f(x)$ at a point near $x = 16.328$ to approximate the value of $f(16.328) = \sqrt{16.328}$. In this case, we know $f(16) = \sqrt{16} = 4$, so this will be our nearby point.

$$\begin{aligned} f'(x) &= \frac{\mathrm{d}}{\mathrm{dx}} x^{1/2} \\ &= \frac{1}{2x^{1/2}}, \end{aligned}$$

so $f'(16) = \frac{1}{2\sqrt{16}} = \frac{1}{8}$, and the tangent line to f at $x = 16$ is

$$y - 4 = \frac{1}{8}(x - 16),$$

or more simply

$$y = \frac{1}{8}x + 2.$$

This tangent line will give us a decent approximation of $f(16.328)$. Our approximation is $y = \frac{1}{8} \cdot 16.328 + 2 = 4.041$.

When compared to the actual value of $\sqrt{16.328} = 4.0408\ldots$, this is a pretty darn good approximation.

33. One side of a right triangle is known to be exactly twelve inches. The angle θ is measured to be π radians with a maximum possible error of ± 0.02 radians.

(a) Use differentials to estimate the maximum error in calculating the length of the opposite side x.

Let $x(\theta) = 12\tan(\theta)$ be the formula for x in terms of θ. To estimate the maximum error in calculating the x when θ varies by ± 0.02 radians, we'll use a tangent line to $x(\theta)$ at $\theta = \frac{\pi}{4}$. Here,

$$x'(\theta) = 12\sec^2(\theta),$$

and $x'(\frac{\pi}{4}) = 12\sec^2(\frac{\pi}{4}) = 6$.

So the tangent line to $x(\theta)$ at $\theta = \frac{\pi}{4}$ (where our independent variable is θ and our dependent variable is x) is

$$x - 12 = 6(\theta - \frac{\pi}{4}).$$

Now, if we look at the x values of our tangent line when $\theta = \frac{\pi}{4} + 0.02$ and $\theta = \frac{\pi}{4} - 0.02$, we find the variance of x is ± 0.48 inches.

(b) Find the maximum percentage error.

We simply divide our estimated error by the true value of $x(\frac{\pi}{4})$:

$$\frac{0.48}{x\left(\frac{\pi}{4}\right)} \cdot 100\% = \frac{0.48}{12} \cdot 100\% = 4\%.$$