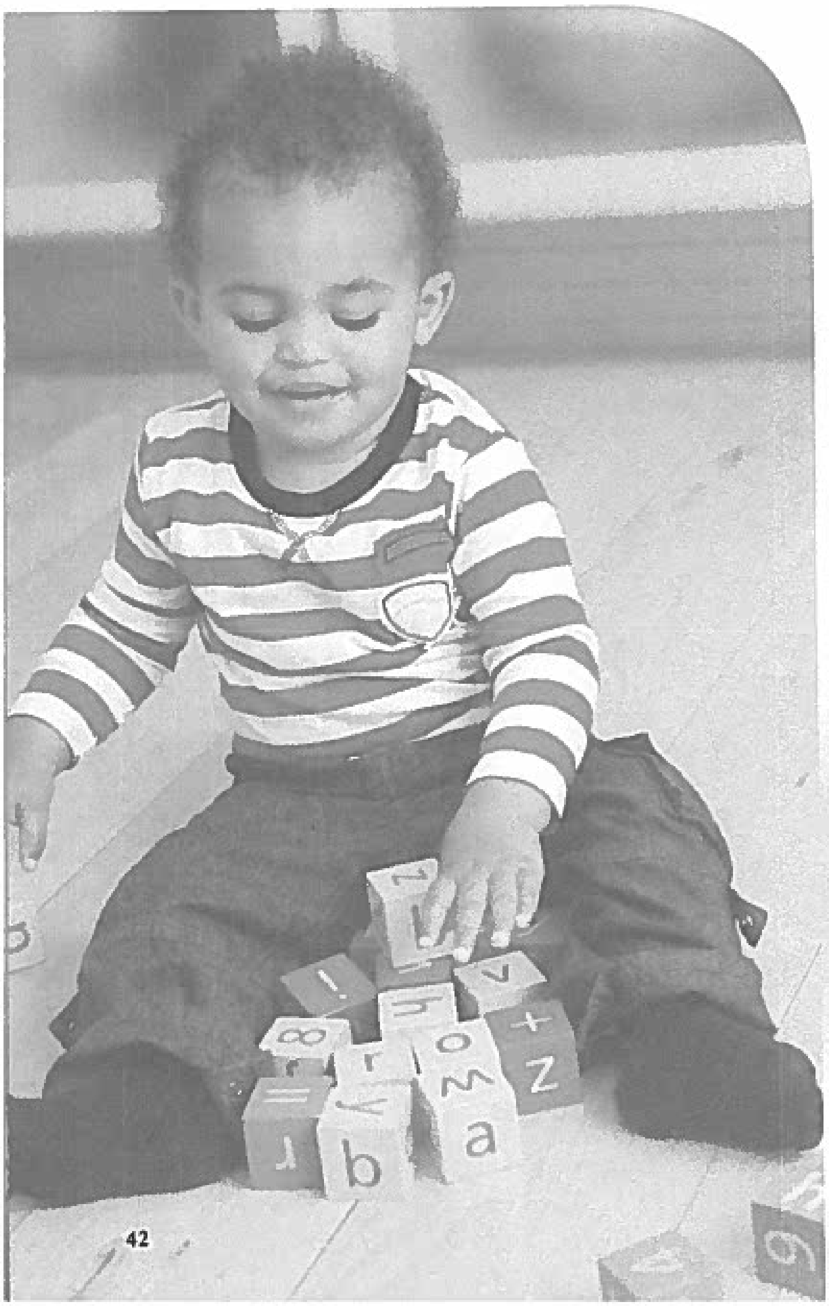
2

Sets



◄ *Children learn how to classify sets, such as shapes and colors, at a very early age.*

What You Will Learn

- Methods to indicate sets, equal sets, and equivalent sets
- Subsets and proper subsets
- Venn diagrams
- Set operations such as complement, intersection, union, difference, and Cartesian product
- Equality of sets
- Applications of sets
- Infinite sets

Why This is Important

A basic human impulse is to sort or classify things. As you will see in this chapter, putting elements into sets helps you order and arrange your world. It allows you to deal with large quantities of information. Set building is a learning tool that helps answer the question, What are the characteristics of this group? For example, when students read a college catalog to determine which courses fulfill their degree requirements, they are looking for courses that are in a particular set. Studying sets is also important because sets underlie other mathematical topics such as logic and abstract algebra.

SECTION 2.1 Set Concepts

▲ *Many different restaurant categories include McDonald's*

Can you think of a few different categories or groups to which the restaurant McDonald's belongs? One way you could categorize McDonald's is as a fast food restaurant. Another way is as a restaurant selling hamburgers. A third way is as a restaurant selling breakfast. In this section, we will discuss ways to sort or classify objects. We will also discuss different methods that can be used to indicate collections of objects.

Why This Is Important Set classifications are important in a range of applications from placing students in courses to classifying stars in the universe.

Profile in Mathematics

Georg Cantor

Georg Cantor (1845–1918), born in St. Petersburg, Russia, is recognized as the founder of set theory. Cantor's creative work in mathematics was nearly lost when his father insisted that he become an engineer rather than a mathematician. His two major books on set theory, *Foundations of General Theory of Aggregates* and *Contributions to the Founding of the Theory of Transfinite Numbers,* were published in 1883 and 1895, respectively. See the Profile in Mathematics on page 85 for more information on Cantor and Leopold Kronecker.

We encounter sets in many different ways every day of our lives. A ***set*** is a collection of objects, which are called ***elements*** or ***members*** of the set. For example, the United States is a collection, or set, of 50 states plus the District of Columbia. The 50 individual states plus the District of Columbia are the members or elements of the set that is called the United States.

A set is ***well defined*** if its contents can be clearly determined. The set of U.S. presidents is a well-defined set because its contents, the presidents, can be named. The set of the three best movies is not a well-defined set because the word *best* is interpreted differently by different people. In this text, we use only well-defined sets.

Three methods are commonly used to indicate a set: (1) description, (2) roster form, and (3) set-builder notation.

The method of indicating a set by ***description*** is illustrated in Example 1.

Example 1 *Description of Sets*

Write a description of the set containing the elements Monday, Tuesday, Wednesday, Thursday, Friday, Saturday, Sunday.

Solution The set is the days of the week. ■

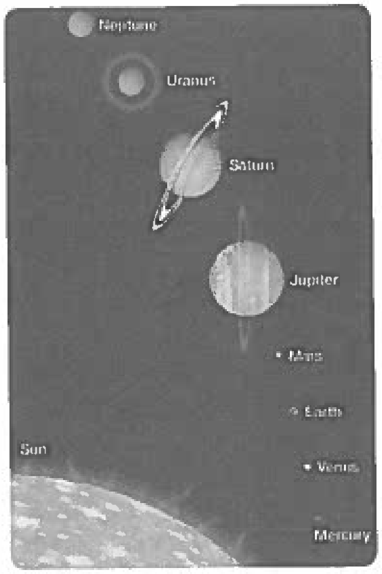
Listing the elements of a set inside a pair of ***braces***, $\{\quad\}$, is called ***roster form***. The braces are an essential part of the notation because they identify the contents as a set. For example, $\{1, 2, 3\}$ is notation for the set whose elements are 1, 2, and 3, but (1, 2, 3) and [1, 2, 3] are not sets because parentheses and brackets do not indicate a set. For a set written in roster form, commas separate the elements of the set. The order in which the elements are listed is not important.

Sets are generally named with capital letters. For example, the name commonly selected for the set of ***natural numbers*** or ***counting numbers*** is N.

Definition: Natural Numbers

$$N = \{1, 2, 3, 4, 5, \ldots\}$$

The three dots after the 5, called an ***ellipsis,*** indicate that the elements in the set continue in the same manner. An ellipsis followed by a last element indicates that the elements continue in the same manner up to and including the last element. This notation is illustrated in Example 2(b).



▲ *The planets of Earth's solar system.*

Example 2 *Roster Form of Sets*

Express the following in roster form.

a) Set A is the set of natural numbers less than 6.

b) Set B is the set of natural numbers less than or equal to 80.

c) Set P is the set of planets in Earth's solar system.

Solution

a) The natural numbers less than 6 are 1, 2, 3, 4, and 5. Thus, set A in roster form is $A = \{1, 2, 3, 4, 5\}$.

b) $B = \{1, 2, 3, 4, \ldots, 80\}$. The 80 after the ellipsis indicates that the elements continue in the same manner up to and including the number 80.

c) $P = \{\text{Mercury, Venus, Earth, Mars, Jupiter, Saturn, Uranus, Neptune}\}$* ■

Example 3 *The Word* Inclusive

Express the following in roster form.

a) The set of natural numbers between 7 and 12.

b) The set of natural numbers between 7 and 12, inclusive.

Solution

a) $A = \{8, 9, 10, 11\}$

b) $B = \{7, 8, 9, 10, 11, 12\}$. Note that the word *inclusive* indicates that the values of 7 and 12 are included in the set. ■

The symbol $\in$, read, "is an element of," is used to indicate membership in a set. In Example 3, because 8 is an element of set A, we write $8 \in A$. This may also be written $8 \in \{8, 9, 10, 11\}$. We may also write $6 \notin A$, meaning that 6 is not an element of set A.

Set-builder notation (sometimes called *set-generator notation*) may be used to symbolize a set. Set-builder notation is frequently used in algebra. The following example illustrates its form.

D	$=$	$\{$	x	$\vert$	Condition(s) $\}$
↑	↑	↑	↑	↑	↑
Set D	is	the set of	all elements x	such that	the condition(s) x must meet in order to be a member of the set.

Consider $E = \{x \mid x \in N \text{ and } x > 10\}$. The statement is read: "Set E is the set of all the elements x such that x is a natural number and x is greater than 10." The conditions that x must meet to be a member of the set are $x \in N$, which means that x must be a natural number, and $x > 10$, which means that x must be greater than 10. The numbers that meet both conditions are the set of natural numbers greater than 10. Set E in roster form is

$$E = \{11, 12, 13, 14, \ldots\}$$

Example 4 *Using Set-Builder Notation*

a) Write set $B = \{1, 2, 3, 4, 5\}$ in set-builder notation.

b) Write, in words, how you would read set B in set-builder notation.

*In August 2006, Pluto was reclassified as a dwarf planet.

Solution

a) Because set B consists of the natural numbers less than 6, we write

$$B = \{x \mid x \in N \text{ and } x < 6\}$$

Another acceptable answer is $B = \{x \mid x \in N \text{ and } x \leq 5\}$.

b) Set B is the set of all elements x such that x is a natural number and x is less than 6. ■

Example 5 *Roster Form to Set-Builder Notation*

a) Write set $C = \{$North America, South America, Europe, Asia, Australia, Africa, Antarctica$\}$ in set-builder notation.

b) Write in words how you would read set C in set-builder notation.

Solution

a) $C = \{x \mid x \text{ is a continent}\}$.

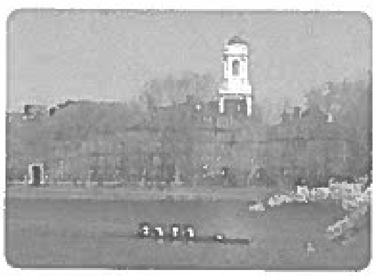
b) Set C is the set of all elements x such that x is a continent. ■

Example 6 *Set-Builder Notation to Roster Form*

Write set $A = \{x \mid x \in N \text{ and } 2 \leq x < 8\}$ in roster form.

Solution $A = \{2, 3, 4, 5, 6, 7\}$ ■

Example 7 *Oldest Colleges in the United States*



▲ *Harvard University*

The table shows the 10 oldest colleges in the United States. Let set C be the set of colleges that are located in Virginia that are among the 10 oldest colleges in the United States. Write set C in roster form.

Ten Oldest Colleges in the United States	State	Year Chartered
Harvard University	Massachusetts	1636
College of William and Mary	Virginia	1692
Yale University	Connecticut	1701
University of Pennsylvania	Pennsylvania	1740
Moravian College	Pennsylvania	1742
Princeton University	New Jersey	1746
Washington and Lee University	Virginia	1749
Columbia University	New York	1754
Brown University	Rhode Island	1764
Rutgers	New Jersey	1766

Source: National Center for Education Statistics

Solution By examining the table, we find that two colleges located in Virginia appear in the table. They are College of William and Mary and Washington and Lee University. Thus, set $C = \{$College of William and Mary, Washington and Lee University$\}$. ■

A set is said to be *finite* if it either contains no elements or the number of elements in the set is a natural number. The set $B = \{2, 4, 6, 8, 10\}$ is a finite set because the number of elements in the set is 5, and 5 is a natural number. A set that is not finite is said to be ***infinite***. The set of counting numbers is one example of an infinite set. Infinite sets are discussed in more detail in Section 2.6.

Another important concept is equality of sets.

Definition: Equal Sets

Set A is **equal** to set B, symbolized by $A = B$, if and only if set A and set B contain exactly the same elements.

For example, if set $A = \{1, 2, 3\}$ and set $B = \{3, 1, 2\}$, then $A = B$ because they contain exactly the same elements. The order of the elements in the set is not important. If two sets are equal, both must contain the same number of elements. The number of elements in a set is called its *cardinal number*.

Definition: Cardinal Number

The **cardinal number** of set A, symbolized by $n(A)$, is the number of elements in set A.

Both set $A = \{1, 2, 3\}$ and set $B = \{\text{England, Brazil, Japan}\}$ have a cardinal number of 3; that is, $n(A) = 3$, and $n(B) = 3$. We can say that set A and set B both have a cardinality of 3.

Two sets are said to be *equivalent* if they contain the same number of elements.

Definition: Equivalent Sets

Set A is **equivalent** to set B if and only if $n(A) = n(B)$.

Any sets that are equal must also be equivalent. Not all sets that are equivalent are equal, however. The sets $D = \{\text{a, b, c}\}$ and $E = \{\text{apple, orange, pear}\}$ are equivalent because both have the same cardinal number, 3. Because the elements differ, however, the sets are not equal.

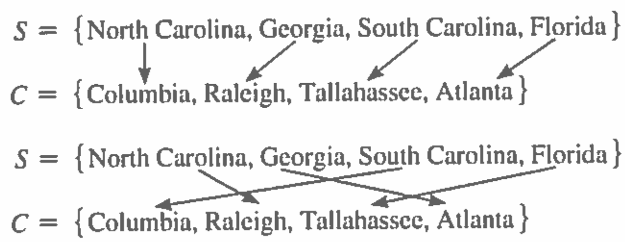
Two sets that are equivalent or have the same cardinality can be placed in ***one-to-one correspondence***. Set A and set B can be placed in one-to-one correspondence if every element of set A can be matched with exactly one element of set B and every element of set B can be matched with exactly one element of set A. For example, there is a one-to-one correspondence between the student names on a class list and the student identification numbers because we can match each student with a student identification number.

Consider set S, states, and set C, state capitals.

$S = \{\text{North Carolina, Georgia, South Carolina, Florida}\}$

$C = \{\text{Columbia, Raleigh, Tallahassee, Atlanta}\}$

Two different one-to-one correspondences for sets S and C follow.



Other one-to-one correspondences between sets S and C are possible. Do you know which capital goes with which state?

Null or Empty Set

Some sets do not contain any elements, such as the set of zebras that live in your house.

Definition: Empty Set
The set that contains no elements is called the **empty set** or **null set** and is symbolized by $\{\ \}$ or $\varnothing$.

Note that $\{\varnothing\}$ is not the empty set. This set contains the element $\varnothing$ and has a cardinality of 1. The set $\{0\}$ is also not the empty set because it contains the element 0. It also has a cardinality of 1.

Example 8 *Natural Number Solutions*

Indicate the set of natural numbers that satisfies the equation $x + 2 = 0$.

Solution The values that satisfy the equation are those natural numbers that make the equation a true statement. Only the number -2 satisfies this equation. Because -2 is not a natural number, the solution set of this equation is $\{\ \}$ or $\varnothing$. ■

Universal Set

Another important set is a *universal set.*

Definition: Universal Set
A **universal set**, symbolized by U, is a set that contains all the elements for any specific discussion.

When a universal set is given, only the elements in the universal set may be considered when working the problem. If, for example, the universal set for a particular problem is defined as $U = \{1, 2, 3, 4, \ldots, 10\}$, then only the natural numbers 1 through 10 may be used in that problem.

SECTION 2.1 Exercises

Warm Up Exercises

In Exercises 1–12, fill in the blank with an appropriate word, phrase, or symbol(s).

1. A collection of objects is called a(n) ________.
2. Three dots placed in a set to show that the set continues in the same manner is called a(n) ________.
3. The three ways a set can be written are ________, ________, and ________.
4. A set that contains no elements or the number of elements in the set is a natural number is called a(n)________ set.
5. A a set that is not finite is called a(n) ________. set.
6. Two sets that contain the same elements are called ________ sets.
7. Two sets that contain the same number of elements are called ________ sets.
8. The number of elements in a set is called the ________ number.
9. The set that contains no elements is called the ________ set.
10. The two ways to indicate an empty set are ________ and ________.
11. A set that contains all the elements for any specific discussion is called a(n) ________ set.
12. Two sets that have the same cardinal number can be placed in a(n) ________ correspondence.

Practice the Skills

In Exercises 13–18, determine whether each set is well defined or not well defined.

13. The set of the best colleges

14. The set of the most interesting courses at your school

15. The set of states that have a common border with Kansas

16. The set of the four states in the United States having the largest population on January 1, 2010

17. The set of astronauts who walked on the moon

▲ *Eugene A. Cernan on the moon*

18. The set of the most interesting teachers at your school

In Exercises 19–24, determine whether each set is finite or infinite.

19. $\{1, 3, 5, 7, \ldots\}$

20. The set of multiples of 4 between 0 and 50

21. The set of odd numbers greater than 25

22. The set of fractions between 1 and 2

23. The set of odd numbers greater than 15

24. The set of apple trees in Gro-More Farms Orchards

In Exercises 25–34, express each set in roster form. You may need to use the Internet or some other reference source.

25. The set of states in the United States whose names begin with the letter M

26. The set of countries in Europe whose names begin with S

27. The set of natural numbers between 10 and 178

28. $C = \{x \mid x + 6 = 10\}$

29. $B = \{x \mid x \in N \text{ and } x \text{ is even}\}$

30. The set of states west of the Mississippi River that have a common border with the state of Florida

31. The set of football players over the age of 70 who are still playing in the National Football League

32. The set of states in the United States that have a common border with the state of Washington

33. $E = \{x \mid x \in N \text{ and } 14 \le x < 85\}$

34. The set of states in the United States that are not in the contiguous 48 states

In Exercises 35–38, use the following table, which shows the attendance, in millions, at the 10 most visited museums in the world in 2008. Let the 10 museums in the list represent the universal set.

Museum	Attendance (in millions)	Location
1. Louvre Museum	8.50	Paris, France
2. British Museum	5.93	London, UK
3. National Gallery of Art	4.96	Washington, DC
4. Tate Modern	4.95	London, UK
5. Metropolitan Museum of Art	4.82	New York, NY
6. Vatican Museums	4.44	Vatican City
7. National Gallery	4.38	London, UK
8. Musee d'Orsay	3.03	Paris, France
9. Musee d'Art Moderne Prado	2.98	Paris, France
10. Museum of Modern Art	2.90	New York, NY

Source: The *Art Newspaper*

Use the list to determine each set in roster form.

35. The set of museums in which the attendance was more than 4.5 million

36. The set of museums in which the attendance was less than 3 million

37. The set of museums in which the attendance was between 2 million and 4 million

38. The set of museums in which the attendance was between 3.5 million and 5.5 million

In Exercises 39–42, use the graph on page 49, which shows iPod sales, in millions, for the years 2003–2008.

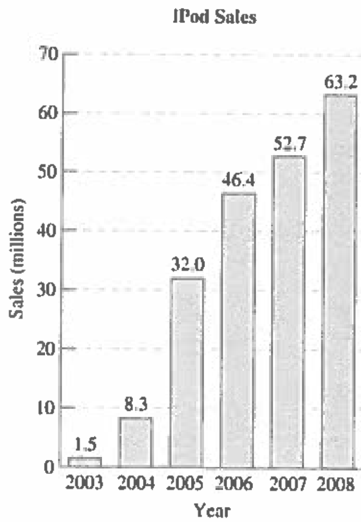
Use the graph to determine each set in roster form.

39. The set of years in which iPod sales were more than 52 million

40. The set of years in which iPod sales were less than 8 million

41. The set of years in which iPod sales were between 8 million and 60 million

42. The set of years in which iPod sales were more than 65 million



In Exercises 43–50, express each set in set-builder notation.

43. $B = \{7, 8, 9, 10, 11, 12, 13, 14\}$

44. $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$

45. $C = \{3, 6, 9, 12, \dots\}$

46. $D = \{5, 10, 15, 20, \dots\}$

47. E is the set of odd natural numbers.

48. A is the set of national holidays in the United States in July.

49. C is the set of months that contain less than 30 days.

50. $F = \{15, 16, 17, \dots, 100\}$

In Exercises 51–58, write a description of each set.

51. $A = \{1, 2, 3, 4, 5, 6, 7\}$

52. $D = \{3, 6, 9, 12, 15, 18, \dots\}$

53. $V = \{a, e, i, o, u\}$

54. $S = \{$Bashful, Doc, Dopey, Grumpy, Happy, Sleepy, Sneezy$\}$

▲ *Snow White and the Seven Dwarfs*

55. $T = \{$oak, maple, elm, pine, ...$\}$

56. $E = \{x \mid x \in N \text{ and } 4 \leq x < 11\}$

57. $S = \{$spring, summer, fall, winter$\}$

58. $B = \{$John Lennon, Ringo Starr, Paul McCartney, George Harrison$\}$

▲ *The Beatles*

In Exercises 59–62, use the following list, which shows the 10 countries with the most cellular subscribers, in millions, as of 2008. Let the 10 countries in the list represent the universal set.

Country	Number of Subscribers
1. China	649.70
2. India	376.12
3. United States	260.00
4. Russia	172.00
5. Brazil	151.90
6. Indonesia	115.60
7. Japan	102.98
8. Germany	101.50
9. Pakistan	91.40
10. United Kingdom	70.00

Source: CIA

Use the list to determine each set in roster form.

59. $\{x \mid x$ is a country with at least 250 million cellular subscribers$\}$

60. $\{x \mid x$ is a country with fewer than 100 million cellular subscribers$\}$

61. $\{x \mid x$ is a country with between 100 million and 200 million cellular subscribers$\}$

62. $\{x \mid x$ is a country with between 250 million and 500 million cellular subscribers$\}$

In Exercises 63–66, use the following graph, which shows the cost of a 30-second commercial during the Super Bowl from 1996 to 2011. Let the 16 years represent the universal set.

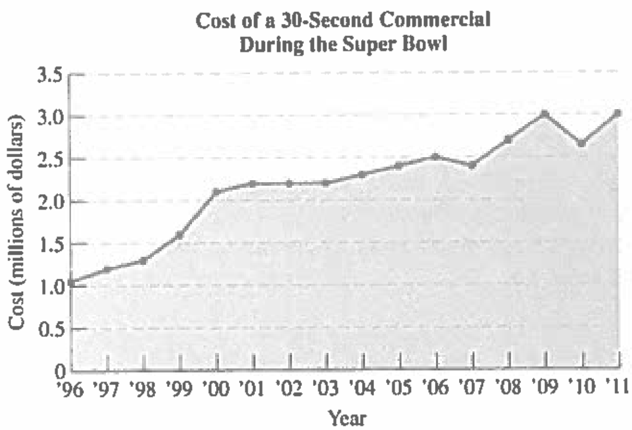
Use the graph to represent each set in roster form.

63. The set of years in which the cost of a Super Bowl commercial was more than \$2.5 million

64. The set of years in which the cost of a Super Bowl commercial was less than \$2.0 million

65. The set of years in which the cost of a Super Bowl commercial was between \$2.0 million and \$2.5 million

66. The set of years in which the cost of a Super Bowl commercial was more than \$2.5 million and less then \$3 million



Source: www.industry.bnet.com

In Exercises 67–74, state whether each statement is true or false. If false, give the reason.

67. $\{e\} \in \{a, e, i, o, u\}$

68. $b \in \{a, b, c, d, e, f\}$

69. $h \in \{a, b, c, d, e, f\}$

70. Mickey Mouse $\in$ {characters created by Walt Disney}

71. $3 \notin \{x \mid x \in N \text{ and } x \text{ is odd}\}$

72. Amazon $\in$ {rivers in the United States}

73. *Titanic* $\in$ {top 10 motion pictures with the greatest revenues}

74. $2 \in \{x \mid x \text{ is an odd natural number}\}$

In Exercises 75–78, for the sets $A = \{2, 4, 6, 8\}$, $B = \{1, 3, 7, 9, 13, 21\}$, $C = \{\ \}$, *and* $D = \{\#, \&, \%, \square, *\}$.

75. Determine $n(A)$.

76. Determine $n(B)$.

77. Determine $n(C)$.

78. Determine $n(D)$.

In Exercises 79–84, determine whether the pairs of sets are equal, equivalent, both, or neither.

79. A = {algebra, geometry, trigonometry},
B = {geometry, trigonometry, algebra}

80. $A = \{7, 9, 10\}$, $B = \{a, b, c\}$

81. A = {grapes, apples, oranges},
B = {grapes, peaches, apples, oranges}

82. A is the set of Siamese cats.
B is the set of cats.

83. A is the set of letters in the word *bank*.
B is the set of letters in the word *post*.

84. A is the set of states.
B is the set of state capitals.

Problem Solving

85. Set-builder notation is often more versatile and efficient than listing a set in roster form. This versatility is illustrated with the following two sets.

$$A = \{x \mid x \in N \text{ and } x > 2\}$$
$$B = \{x \mid x > 2\}$$

a) Write a description of set A and set B.

b) Explain the difference between set A and set B. (*Hint:* Is $4\frac{1}{2} \in A$? Is $4\frac{1}{2} \in B$?)

c) Write set A in roster form.

d) Can set B be written in roster form? Explain your answer.

86. Consider sets A and B below

$$A = \{x \mid 2 < x \leq 5 \text{ and } x \in N\}$$

and

$$B = \{x \mid 2 < x \leq 5\}$$

a) Write a description of set A and set B.

b) Explain the difference between set A and set B.

c) Write set A in roster form.

d) Can set B be written in roster form? Explain your answer.

A cardinal number answers the question "How many?" An ***ordinal number*** *describes the relative position that an element occupies. For example, Molly's desk is the third desk from the aisle.*

In Exercises 87–90, determine whether the number used is a cardinal number or an ordinal number.

87. J. K. Rowling has written 7 Harry Potter books.

▲ *J. K. Rowling*

88. Study the chart on page 25 in the book.

89. Lincoln was the sixteenth president of the United States.

90. Emily paid $35 for her new blouse.

91. Describe three sets of which you are a member.

92. Describe three sets that have no members.

93. Write a short paragraph explaining why the universal set and the empty set are necessary in the study of sets.

Challenge Problem/Group Activity

94. a) In a given exercise, a universal set is not specified, but we know that actor Orlando Bloom is a member of the universal set. Describe five different possible universal sets of which Orlando Bloom is a member.

b) Write a description of one set that includes all the universal sets in part (a).

Internet/Research Activity

95. Georg Cantor is recognized as the founder and a leader in the development of set theory. Do research and write a paper on his life and his contributions to set theory and to the field of mathematics. References include history of mathematics books, encyclopedias, and the Internet.

SECTION 2.2 Subsets

▲ *The set of intercollegiate sports includes basketball.*

Consider the following sets. Set $A = \{$baseball, basketball, hockey$\}$. Set $B = \{$baseball, football, basketball, hockey, softball$\}$. Note that each element of set A is also an element of set B. In this section, we will discuss how to illustrate the relationship between two sets, A and B, when each element of set A is also an element of set B.

Why ***This Is Important*** The relationship between sets is important throughout life. For example, to gain a promotion at work, you may need to fulfill different sets of criteria.

In our complex world, we often break larger sets into smaller, more manageable sets, called *subsets*. For example, consider the set of people in your class. Suppose we categorize the set of people in your class according to the first letter of their last name (the A's, B's, C's, etc.). When we do so, each of these sets may be considered a subset of the original set. Each of these subsets can be separated further. For example, the set of people whose last name begins with the letter A can be categorized as either male or female or by their age. Each of these collections of people is also a subset. A given set may have many different subsets.

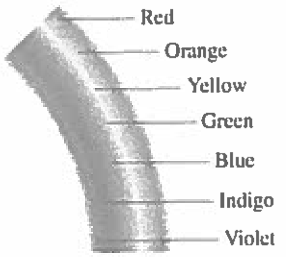
Definition: Subset

Set A is a **subset** of set B, symbolized by $A \subseteq B$, if and only if all the elements of set A are also elements of set B.

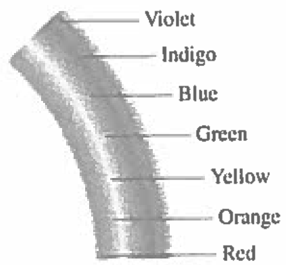
The symbol $A \subseteq B$ indicates that "set A is a subset of set B." The symbol $\not\subseteq$ is used to indicate "is not a subset." Thus, $A \not\subseteq B$ indicates that set A is not a subset of set B. *To show that set A is not a subset of set B, we must find at least one element of set A that is not an element of set B.*

Did You Know?

Rainbows



Colors of primary rainbow



Colors of secondary rainbow

Most rainbows we see are primary rainbows, but there are rare moments when a second, fainter rainbow can be seen behind the first. In this secondary rainbow, the light pattern has been reversed. Both rainbows contain the same set of colors, so each set of colors is a subset of the other.

Example 1 *A Subset?*

Determine whether set A is a subset of set B.

a) $A = \{\text{rain, snow, sleet}\}$
   $B = \{\text{rain, snow, sleet, hail}\}$

b) $A = \{\text{q, r, s, t}\}$
   $B = \{\text{q, r}\}$

c) $A = \{x \mid x \text{ is a yellow fruit}\}$
   $B = \{x \mid x \text{ is a red fruit}\}$

d) $A = \{\text{vanilla, chocolate, rocky road}\}$
   $B = \{\text{chocolate, vanilla, rocky road}\}$

Solution

a) All the elements of set A are contained in set B, so $A \subseteq B$.

b) The elements s and t are in set A but not in set B, so $A \not\subseteq B$ (A is not a subset of B). In this example, however, all the elements of set B are contained in set A; therefore, $B \subseteq A$.

c) There are fruits, such as bananas, that are in set A that are not in set B, so $A \not\subseteq B$.

d) All the elements of set A are contained in set B, so $A \subseteq B$. Note also that $B \subseteq A$. In fact, set A = set B. ■

Proper Subsets

Definition: Proper Subset

Set A is a **proper subset** of set B, symbolized by $A \subset B$, if and only if all the elements of set A are elements of set B and set $A \neq$ set B (that is, set B must contain at least one element not in set A).

Consider the sets $A = \{\text{red, blue, yellow}\}$ and $B = \{\text{red, orange, yellow, green, blue, violet}\}$. Set A is a *subset* of set B, $A \subseteq B$, because every element of set A is also an element of set B. Set A is also a *proper subset* of set B, $A \subset B$, because set A and set B are not equal. Now consider $C = \{\text{car, bus, train}\}$ and $D = \{\text{train, car, bus}\}$. Set C is a subset of set D, $C \subseteq D$, because every element of set C is also an element of set D. Set C, however, is not a proper subset of set D, $C \not\subset D$, because set C and set D are equal sets.

Example 2 *A Proper Subset?*

Determine whether set A is a proper subset of set B.

a) $A = \{\text{jazz, pop, hip hop}\}$
   $B = \{\text{classical, jazz, pop, rap, hip hop}\}$

b) $A = \{a, b, c, d\}$ $B = \{a, c, b, d\}$

Solution

a) All the elements of set A are contained in set B, and sets A and B are not equal; thus, $A \subset B$.

b) Set A = set B, so $A \not\subset B$. (However, $A \subseteq B$.) ■

Every set is a subset of itself, but no set is a proper subset of itself. For all sets A, $A \subseteq A$, but $A \not\subset A$. For example, if $A = \{1, 2, 3\}$, then $A \subseteq A$ because every element of set A is contained in set A, but $A \not\subset A$ because set A = set A.

MATHEMATICS TODAY

The Ladder of Life

In biology, the science of classifying all living things is called *taxonomy*. More than 2000 years ago, Aristotle formalized animal classification with his "ladder of life": higher animals, lower animals, higher plants, lower plants. Today, living organisms are classified into six kingdoms (or sets) called animalia, plantae, archaea, eubacteria, fungi, and protista. Even more general groupings of living things are made according to shared characteristics. The groupings, from most general to most specific, are kingdom, phylum, class, order, family, genus, and species. For example, a zebra, *Equus burchelli*, is a member of the genus *Equus*, as is the horse, *Equus caballus*. Both the zebra and the horse are members of the universal set called the kingdom of animals and the same family, Equidae; they are members of different species (*E. burchelli* and *E. caballus*), however.

Why This Is Important Scientists use sets to classify and catorgize animals, plants, and all forms of life.

Let $A = \{\ \}$ and $B = \{1, 2, 3, 4\}$. Is $A \subseteq B$? To show $A \not\subseteq B$, you must find at least one element of set A that is not an element of set B. Because this cannot be done, $A \subseteq B$ must be true. Using the same reasoning, we can show that *the empty set is a subset of every set, including itself.*

Example 3 ***Element or Subset?***

Determine whether the following are true or false.

a) $3 \in \{3, 4, 5\}$
b) $\{3\} \in \{3, 4, 5\}$
c) $\{3\} \in \{\{3\}, \{4\}, \{5\}\}$
d) $\{3\} \subseteq \{3, 4, 5\}$
e) $3 \subseteq \{3, 4, 5\}$
f) $\{\ \} \subseteq \{3, 4, 5\}$

Solution

a) $3 \in \{3, 4, 5\}$ is a true statement because 3 is an element of the set $\{3, 4, 5\}$.
b) $\{3\} \in \{3, 4, 5\}$ is a false statement because $\{3\}$ is a set, and the set $\{3\}$ is not an element of the set $\{3, 4, 5\}$.
c) $\{3\} \in \{\{3\}, \{4\}, \{5\}\}$ is a true statement because $\{3\}$ is an element in the set. The elements of the set $\{\{3\}, \{4\}, \{5\}\}$ are themselves sets.
d) $\{3\} \subseteq \{3, 4, 5\}$ is a true statement because every element of the first set is an element of the second set.
e) $3 \subseteq \{3, 4, 5\}$ is a false statement because the 3 is not in braces, so it is not a set and thus cannot be a subset. The 3 is an element of the set as indicated in part (a).
f) $\{\ \} \subseteq \{3, 4, 5\}$ is a true statement because the empty set is a subset of every set. ■

Number of Subsets

How many distinct subsets can be made from a given set? The empty set has no elements and has exactly one subset, the empty set. A set with one element has two subsets. A set with two elements has four subsets. A set with three elements has eight subsets. This information is illustrated in Table 2.1.

Table 2.1 Number of Subsets

Set	Subsets	Number of Subsets
$\{\ \}$	$\{\ \}$	$1 = 2^0$
$\{a\}$	$\{a\}$ $\{\ \}$	$2 = 2^1$
$\{a, b\}$	$\{a, b\}$ $\{a\}, \{b\}$ $\{\ \}$	$4 = 2 \times 2 = 2^2$
$\{a, b, c\}$	$\{a, b, c\}$ $\{a, b\}, \{a, c\}, \{b, c\}$ $\{a\}, \{b\}, \{c\}$ $\{\ \}$	$8 = 2 \times 2 \times 2 = 2^3$

By continuing this table with larger and larger sets, we can develop a general expression for finding the number of distinct subsets that can be made from any given set.

Number of Distinct Subsets

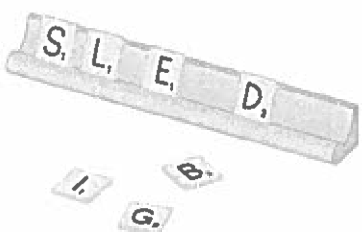
The **number of distinct subsets** of a finite set A is 2^n, where n is the number of elements in set A.

Every set is a subset of itself, but no set is a proper subset of itself. Thus, the number of proper subsets will always be one less than the number of subsets that can be made from any given set. We summarize this concept in the following expression.

Number of Distinct Proper Subsets

The **number of distinct proper subsets** of a finite set A is $2^n - 1$, where n is the number of elements in set A.

Example 4 *Distinct Subsets*



a) Determine the number of distinct subsets for the set $\{S, L, E, D\}$.

b) List all the distinct subsets for the set $\{S, L, E, D\}$.

c) How many of the distinct subsets are proper subsets?

Solution

a) Since the number of elements in the set is 4, the number of distinct subsets is $2^4 = 2 \times 2 \times 2 \times 2 = 16$.

b)

$\{S, L, E, D\}$	$\{S, L, E\}$	$\{S, L\}$	$\{S\}$	$\{ \}$
	$\{S, L, D\}$	$\{S, E\}$	$\{L\}$	
	$\{S, E, D\}$	$\{S, D\}$	$\{E\}$	
	$\{L, E, D\}$	$\{L, E\}$	$\{D\}$	
		$\{L, D\}$		
		$\{E, D\}$		

c) There are 15 proper subsets. Every subset except $\{S, L, E, D\}$ is a proper subset. ■

Example 5 *Car Options*

Brigette Martineau is ordering a new car. She can order some, all, or none of the following options: power windows, MP3 player port, leather interior, alarm system, sun roof, and navigation system. How many different variations of the set of options are possible?

Solution Brigette can order the car with no options, any one option, any two options, any three options, and so on, up to six options. One technique used in problem solving is to consider similar problems that you have solved previously. If you think about this problem, you will realize that it is the same as asking how many distinct subsets can be made from a set with six elements. The number of different variations of the set of options is the same as the number of possible subsets of a set that has six elements. There are 2^6, or 64, possible subsets of a set with six elements. Thus, there are 64 possible variations of the set of options for the car. ■

SECTION 2.2 Exercises

Warm Up Exercises

In Exercises 1–4, fill in the blank with an appropriate word, phrase, or symbol(s).

1. If all the elements of set A are also elements of set B, then set A is a(n) ________ of set B.
2. If all the elements of set A are also elements of set B, and set $A \neq$ set B, then set A is a (n) ________ subset of set B.
3. The expression for determining the number of distinct subsets for a set with n distinct elements is ________.
4. The expression for determining the number of distinct proper subsets for a set with n distinct elements is ________.

Practice the Skills

In Exercises 5–26, answer true or false. If false, give the reason.

5. {book} ⊆ {magazine, newspaper, book}
6. {Italy} ⊆ {Italy, Spain, France, Switzerland, Austria}
7. {McIntosh, Red Delicious} ⊆ {Empire, Gala, Cortland, Red Delicious}
8. {pepper, salt} ⊆ {salt, butter, mayonnaise}
9. {motorboat, kayak} ⊂ {kayak, fishing boat, motorboat, sailboat}
10. {polar bear, tiger, lion} ⊂ {tiger, lion, polar bear, penguin}
11. $\{4, 2, 7\} \subset \{4, 7, 2\}$
12. {c, a, r, t} ⊂ {t, r, a, c}
13. Xbox 360 ∈ {PSIII, Wii, Xbox 360}
14. LaGuardia ∈ {JFK, LaGuardia, Newark}
15. {swimming} ∈ {sailing, water skiing, swimming}
16. $\{\ \} \in \{1, 3, 5, 7\}$
17. $5 \notin \{2, 4, 6\}$
18. { } ⊆ {table, chair, sofa}
19. {red} ⊂ {red, blue, green}
20. $\{3, 5, 9\} \not\subset \{3, 9, 5\}$
21. $\{\ \} = \{\varnothing\}$
22. $\varnothing = \{\ \}$
23. $\{0\} = \varnothing$
24. $\{\ \} \subseteq \{\ \}$
25. $0 = \{\ \}$
26. $\{1\} \in \{\{1\}, \{2\}, \{3\}\}$

In Exercises 27–34, determine whether $A = B$, $A \subseteq B$, $B \subseteq A$, $A \subset B$, $B \subset A$, or if none of these applies. (There may be more than one answer.)

27. A = {penny, nickel, dime, quarter}
    B = {penny, quarter}
28. $A = \{x \mid x \in N \text{ and } x < 6\}$
    $B = \{x \mid x \in N \text{ and } 1 \leq x \leq 5\}$
29. Set A is the set of states that border the Atlantic Ocean. Set B is the set of states east of the Mississippi River.
30. $A = \{1, 3, 5, 7, 9\}$
    $B = \{3, 9, 5, 7, 6\}$
31. $A = \{x \mid x$ is a brand of soft drink$\}$
    B = {A & W, Coca-Cola, Dr Pepper, Mountain Dew}

32. $A = \{x \mid x$ is a sport that uses a ball$\}$
    B = {basketball, soccer, tennis}
33. Set A is the set of natural numbers between 2 and 7. Set B is the set of natural numbers greater than 2 and less than 7.
34. Set A is the set of all cars manufactured by General Motors. Set B is the set of sports cars manufactured by General Motors.

In Exercises 35–38, list all the subsets of the sets given.

35. $D = \varnothing$
36. $A = \{\bigcirc\}$
37. B = {cow, horse}

38. $C = \{\text{steak, pork, chicken}\}$

Problem Solving

39. For set $A = \{a, b, c, d\}$,

a) list all the subsets of set A.

b) state which of the subsets in part (a) are not proper subsets of set A.

40. A set contains nine elements.

a) How many subsets does it have?

b) How many proper subsets does it have?

In Exercises 41–52, if the statement is true for all sets A and B, write "true." If it is not true for all sets A and B, write "false." Assume that $A \neq \emptyset$, $U \neq \emptyset$, and $A \subset U$.

41. If $A \subseteq B$, then $A \subset B$.

42. If $A \subset B$, then $A \subseteq B$.

43. $A \subseteq A$

44. $A \subset A$

45. $\emptyset \subset A$

46. $\emptyset \subseteq A$

47. $A \subseteq U$

48. $\emptyset \subset \emptyset$

49. $\emptyset \subset U$

50. $U \subseteq \emptyset$

51. $\emptyset \subseteq \emptyset$

52. $U \subset \emptyset$

53. ***Ordering a Pizza*** Jasmine Sullivan is ordering a pizza at Domino's Pizza. She can add any of the following toppings: olives, pepperoni, sausage, onions, green peppers, mushrooms, anchovies, and ham. How many different variations of the pizza and toppings can be made?

54. ***Building a House*** The Jacobsens are planning to build a house in a new development. They can either build the base model offered by the builder or add any of the following options: deck, hot tub, security system, hardwood flooring. How many different variations of the house are possible?

55. ***Salad Toppings*** Donald Wheeler is ordering a salad at a Ruby Tuesday restaurant. He can purchase a salad consisting of just lettuce, or he can add any of the following items: cucumber, onion, tomato, carrot, green pepper, olive, mushroom. How many different variations of a salad are possible?

56. ***Telephone Features*** A customer with Verizon can order telephone service with some, all, or none of the following features: call waiting, call forwarding, caller identification, three-way calling, voice mail, fax line. How many different variations of the set of features are possible?

57. If $E \subseteq F$ and $F \subseteq E$, what other relationship exists between E and F?

58. How can you determine whether the set of boys is equivalent to the set of girls at a roller-skating rink?

59. For the set $D = \{a, b, c\}$

a) is a an element of set D?

b) is c a subset of set D?

c) is $\{a, b\}$ a subset of set D?

Challenge Problem/Group Activity

60. ***Hospital Expansion*** A hospital has four members on the board of directors: Arnold, Benitez, Cathy, and Dominique.

a) When the members vote on whether to add a wing to the hospital, how many different ways can they vote (abstentions are not allowed)? For example, Arnold—yes, Benitez—no, Cathy—no, and Dominique—yes is one of the many possibilities.

b) Make a listing of all the possible outcomes of the vote. For example, the vote described in part (a) could be represented as (YNNY).

c) How many of the outcomes given in part (b) would result in a majority supporting the addition of a wing to the hospital? That is, how many of the outcomes have three or more Y's?

Recreational Mathematics

61. How many elements must a set have if the number of proper subsets of the set is $\frac{1}{2}$ of the total number of subsets of the set?

62. If $A \subset B$ and $B \subset C$, must $A \subset C$?

63. If $A \subset B$ and $B \subseteq C$, must $A \subset C$?

64. If $A \subseteq B$ and $B \subseteq C$, must $A \subset C$?

Internet/Research Activity

65. On page 53, we discussed the ladder of life. Do research and indicate all the different classifications in the Linnaean system, from most general to the most specific, in which a koala belongs.

SECTION 2.3 Venn Diagrams and Set Operations

▲ *Some Laptops have A 14-inch display, some laptops have 4 GB of memory, and some laptops have a 14-inch display and 4 GB of memory.*

Suppose you go to a store to purchase a new laptop and tell a computer salesperson that you wish to purchase a laptop with a 14-inch display *and* 4 GB of memory. The salesperson was a bit distracted and thought you said you wanted to purchase a laptop with a 14-inch display *or* 4 GB of memory. Which laptops are in the set of laptops with a 14-inch display *and* 4 GB of memory? Which laptops are in the set of laptops with a 14-inch display *or* 4 GB of memory? These two questions are quite different. The first involves laptops joined by the word *and*. The second involves laptops joined by the word *or*. In this section, you will learn how to illustrate these and other set relationships.

Why *This is Important* Words such as *and* and *or* have important meaning in a variety of everyday applications, such as ordering from a menu or understanding the meaning of a legal document.

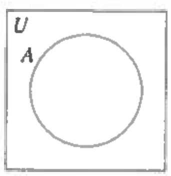


Figure 2.1

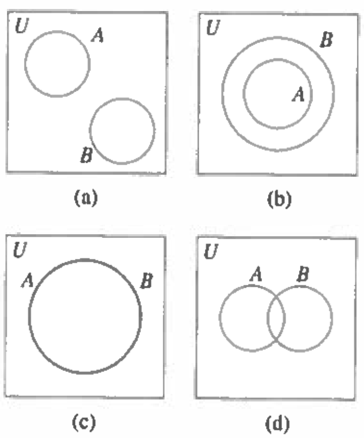


Figure 2.2

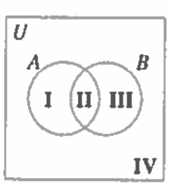


Figure 2.3

A useful technique for illustrating set relationships is the Venn diagram, named for English mathematician John Venn (1834–1923). Venn invented the diagrams and used them to illustrate ideas in his text on symbolic logic, published in 1881.

In a Venn diagram, a rectangle usually represents the universal set, U. The items inside the rectangle may be divided into subsets of the universal set. The subsets are usually represented by circles. In Fig. 2.1, the circle labeled A represents set A, which is a subset of the universal set.

Two sets may be represented in a Venn diagram in any of four different ways, as shown in Fig. 2.2. Two sets A and B are ***disjoint*** when they have no elements in common. Two disjoint sets A and B are illustrated in Fig. 2.2(a). If set A is a proper subset of set B, $A \subset B$, the two sets may be illustrated as in Fig. 2.2(b). If set A contains exactly the same elements as set B, that is, $A = B$, the two sets may be illustrated as in Fig. 2.2(c). Two sets A and B with some elements in common are shown in Fig. 2.2(d), which is regarded as the most general form of a Venn diagram.

If we label the regions of the diagram in Fig. 2.2(d) using I, II, III, and IV, we can illustrate the four possible cases with this one diagram, Fig. 2.3.

CASE 1: DISJOINT SETS When sets A and B are disjoint, they have no elements in common. Therefore, region II of Fig. 2.3 is empty.

CASE 2: SUBSETS When $A \subseteq B$, every element of set A is also an element of set B. Thus, there can be no elements in region I of Fig. 2.3. If $B \subseteq A$, however, then region III of Fig. 2.3 is empty.

CASE 3: EQUAL SETS When set A = set B, all the elements of set A are elements of set B and all the elements of set B are elements of set A. Thus, regions I and III of Fig. 2.3 are empty.

CASE 4: OVERLAPPING SETS When sets A and B have elements in common, those elements are in region II of Fig. 2.3. The elements that belong to set A but not to set B are in region I. The elements that belong to set B but not to set A are in region III.

In each of the four cases, any element belonging to the universal set but not belonging to set A or set B is placed in region IV.

Next we introduce set operations. Venn diagrams will be helpful in understanding set operations. The basic operations of arithmetic are $+$, $-$, $\times$, and $\div$. When we see these symbols, we know what procedure to follow to determine the answer. Some of the operations in set theory are $'$, $\cap$, $\cup$, $-$, and $\times$. They represent complement, intersection, union, difference, and Cartesian product, respectively.

Complement

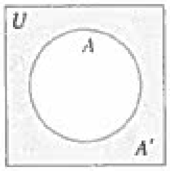


Figure 2.4

Definition: Complement

The **complement** of set A, symbolized by A', is the set of all the elements in the universal set that are not in set A.

In Fig. 2.4, the shaded region outside set A within the universal set represents the complement of set A, or A'.

Example 1 *A Set and Its Complement*

Given

$$U = \{1, 2, 3, 4, 5, 6, 7, 8\} \text{ and } A = \{1, 3, 4\}$$

find A' and illustrate the relationship among sets U, A, and A' in a Venn diagram.

Solution The elements in U that are not in set A are 2, 5, 6, 7, 8. Thus, $A' = \{2, 5, 6, 7, 8\}$. The Venn diagram is illustrated in Fig. 2.5. ■

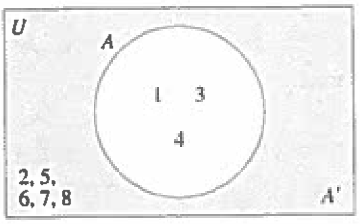
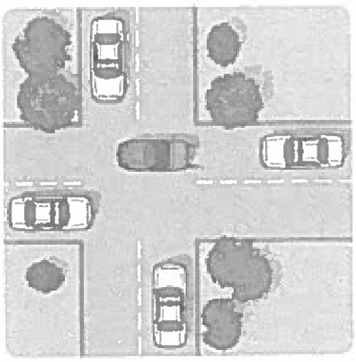


Figure 2.5

Intersection



The word *intersection* brings to mind the area common to two crossing streets. The red car in the figure is in the intersection of the two streets. The set operation intersection is defined as follows.

Definition: Intersection

The **intersection** of sets A and B, symbolized by $A \cap B$, is the set containing all the elements that are common to both set A and set B.

The shaded region, region II, in Fig. 2.6 represents the intersection of sets A and B.

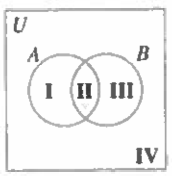


Figure 2.6

Example 2 *Sets with Overlapping Regions*

Let the universal set, U, represent the 50 states in the United States. Let set A represent the set of states with a population of more than 10 million people as of 2009. Let set B represent the set of states that have at least one city with a population of more than 1 million people, as of 2009 (see the table). Draw a Venn diagram illustrating the relationship between set A and set B.

States with a Population of More Than 10 Million People	States with at Least One City with a Population of More Than 1 Million People
California	California
Texas	Texas
New York	New York
Florida	Illinois
Illinois	Pennsylvania
Pennsylvania	Arizona
Ohio	

Source: Bureau of the U.S. Census

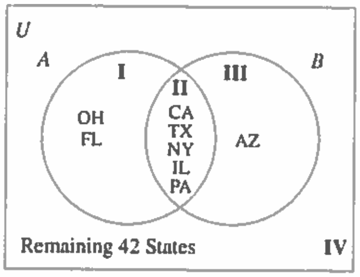


Figure 2.7

Solution First determine the intersection of sets A and B. The states common to both sets are California, Texas, New York, Illinois, and Pennsylvania. Therefore,

$$A \cap B = \{\text{California, Texas, New York, Illinois, Pennsylvania}\}$$

Place these elements in region II of Fig. 2.7. Complete region I by determining the elements in set A that have not been placed in region II. Therefore, Ohio and Florida are placed in region I. Complete region III by determining the elements in set B that have not been placed in region II. Thus, Arizona is placed in region III. Finally, place those elements in U that are not in either set within the rectangle but are outside both circles. This group includes the remaining 42 states, which are placed in region IV. ■

Example 3 *The Intersection of Sets*

Given

$$U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$$
$$A = \{1, 2, 3, 8\}$$
$$B = \{1, 3, 6, 7, 8\}$$
$$C = \{\ \}$$

find

a) $A \cap B$. b) $A \cap C$. c) $A' \cap B$. d) $(A \cap B)'$.

Solution

a) $A \cap B = \{1, 2, 3, 8\} \cap \{1, 3, 6, 7, 8\} = \{1, 3, 8\}$. The elements common to both set A and set B are 1, 3, and 8.

b) $A \cap C = \{1, 2, 3, 8\} \cap \{\ \} = \{\ \}$. There are no elements common to both set A and set C.

c) To determine $A' \cap B$, we must first determine A'.

$$A' = \{4, 5, 6, 7, 9, 10\}$$
$$A' \cap B = \{4, 5, 6, 7, 9, 10\} \cap \{1, 3, 6, 7, 8\} = \{6, 7\}$$

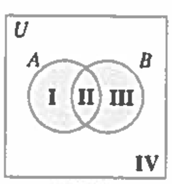
d) To find $(A \cap B)'$, first determine $A \cap B$.

$$A \cap B = \{1, 3, 8\} \text{ from part (a)}$$
$$(A \cap B)' = \{1, 3, 8\}' = \{2, 4, 5, 6, 7, 9, 10\}$$

■

Union

The word *union* means to unite or join together, as in marriage, and that is exactly what is done when we perform the operation of union.



$A \cup B$

Figure 2.8

Definition: Union

The **union** of set A and set B, symbolized by $A \cup B$, is the set containing all the elements that are members of set A or of set B (or of both sets).

The three shaded regions of Fig. 2.8, regions I, II, and III, together represent the union of sets A and B. If an element is common to both sets, it is listed only once in the union of the sets.

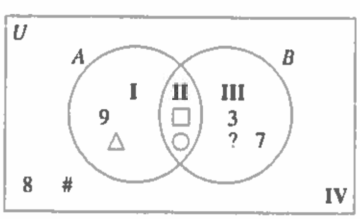


Figure 2.9

Example 4 *Determining Sets from a Venn Diagram*

Use the Venn diagram in Fig. 2.9 to determine the following sets.

a) U b) A c) B' d) $A \cap B$
e) $A \cup B$ f) $(A \cup B)'$ g) $n(A \cup B)$

Solution

a) The universal set consists of all the elements within the rectangle, that is, the elements in regions I, II, III, and IV. Thus, $U = \{9, \triangle, \square, \bigcirc, 3, 7, ?, \#, 8\}$.

b) Set A consists of the elements in regions I and II. Thus, $A = \{9, \triangle, \square, \bigcirc\}$.

c) B' consists of the elements outside set B, or the elements in regions I and IV. Thus, $B' = \{9, \triangle, \#, 8\}$.

d) $A \cap B$ consists of the elements that belong to both set A and set B (region II). Thus, $A \cap B = \{\square, \bigcirc\}$.

e) $A \cup B$ consists of the elements that belong to set A or set B (regions I, II, or III). Thus, $A \cup B = \{9, \triangle, \square, \bigcirc, 3, 7, ?\}$.

f) $(A \cup B)'$ consists of the elements in U that are not in $A \cup B$. Thus, $(A \cup B)' = \{\#, 8\}$.

g) $n(A \cup B)$ represents the *number of elements* in the union of sets A and B. Thus, $n(A \cup B) = 7$, as there are seven elements in the union of sets A and B. ■

Example 5 *The Union of Sets*

Given

$$U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$$
$$A = \{1, 2, 4, 6\}$$
$$B = \{1, 3, 6, 7, 9\}$$
$$C = \{\ \}$$

determine each of the following.

a) $A \cup B$ b) $A \cup C$ c) $A' \cup B$ d) $(A \cup B)'$

Solution

a) $A \cup B = \{1, 2, 4, 6\} \cup \{1, 3, 6, 7, 9\} = \{1, 2, 3, 4, 6, 7, 9\}$

b) $A \cup C = \{1, 2, 4, 6\} \cup \{\ \} = \{1, 2, 4, 6\}$. Note that $A \cup C = A$.

c) To determine $A' \cup B$, we must determine A'.

$$A' = \{3, 5, 7, 8, 9, 10\}$$
$$A' \cup B = \{3, 5, 7, 8, 9, 10\} \cup \{1, 3, 6, 7, 9\}$$
$$= \{1, 3, 5, 6, 7, 8, 9, 10\}$$

d) Determine $(A \cup B)'$ by first determining $A \cup B$, and then find the complement of $A \cup B$.

$$A \cup B = \{1, 2, 3, 4, 6, 7, 9\} \text{ from part (a)}$$
$$(A \cup B)' = \{1, 2, 3, 4, 6, 7, 9\}' = \{5, 8, 10\}$$ ■

Example 6 *Union and Intersection*

Given

$$U = \{a, b, c, d, e, f, g\}$$
$$A = \{a, b, c, g\}$$
$$B = \{a, c, d, e\}$$
$$C = \{b, e, f\}$$

determine each of the following.

a) $(A \cup B) \cap (A \cup C)$ b) $(A \cup B) \cap C'$ c) $A' \cap B'$

Solution

a) $(A \cup B) \cap (A \cup C) = \{a, b, c, d, e, g\} \cap \{a, b, e, f, g\}$
$= \{a, b, e, g\}$

b) $(A \cup B) \cap C' = \{a, b, c, d, e, g\} \cap \{a, c, d, g\}$
$= \{a, c, d, g\}$

c) $A' \cap B' = \{c, d, f\} \cap \{b, f, g\}$
$= \{f\}$ ■

The Meaning of *and* and *or*

The words *and* and *or* are very important in many areas of mathematics. We use these words in several chapters in this book, including Ch.12, Probability. The word ***and*** is generally interpreted to mean ***intersection***, whereas ***or*** is generally interpreted to mean ***union***. Suppose $A = \{1, 2, 3, 5, 6, 8\}$ and $B = \{1, 3, 4, 7, 9, 10\}$. The elements that belong to set A *and* set B are 1 and 3. These are the elements in the intersection of the sets. The elements that belong to set A *or* set B are 1, 2, 3, 4, 5, 6, 7, 8, 9, and 10. These are the elements in the union of the sets.

The Relationship Between $n(A \cup B)$, $n(A)$, $n(B)$, and $n(A \cap B)$

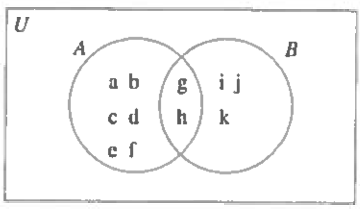


Figure 2.10

Having looked at unions and intersections, we can now determine a relationship between $n(A \cup B)$, $n(A)$, $n(B)$, and $n(A \cap B)$. Suppose set A has eight elements, set B has five elements, and $A \cap B$ has two elements. How many elements are in $A \cup B$? Let's make up some arbitrary sets that meet the criteria specified and draw a Venn diagram. If we let set $A = \{a, b, c, d, e, f, g, h\}$, then set B must contain five elements, two of which are also in set A. Let set $B = \{g, h, i, j, k\}$. We construct a Venn diagram by filling in the intersection first, as shown in Fig. 2.10. The number of elements in $A \cup B$ is 11. The elements g and h are in both sets, and if we add $n(A) + n(B)$, we are counting these elements twice.

To find the number of elements in the union of sets A and B, we can add the number of elements in sets A and B and then subtract the number of elements common to both sets.

The Number of Elements in $A \cup B$

For any finite sets A and B,

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

Example 7 *How Many Visitors Speak Spanish or French?*

The results of a survey of visitors at the Grand Canyon showed that 25 speak Spanish, 14 speak French, and 4 speak both Spanish and French. How many speak Spanish or French?

Solution If we let set A be the set of visitors who speak Spanish and let set B be the set of visitors who speak French, then we need to determine $n(A \cup B)$. We can use the above formula to find $n(A \cup B)$.

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$
$$n(A \cup B) = 25 + 14 - 4$$
$$= 35$$

Thus, 35 of the visitors surveyed speak either Spanish or French. ■

Example 8 *The Number of Elements in Set A*

Of the homes listed for sale with RE/MAX, 39 have either a three-car garage or a fireplace, 31 have a fireplace, and 18 have both a three-car garage and a fireplace. How many of these homes have a three-car garage?

Solution If we let set A be the set of homes with a three-car garage and set B be the set of homes with a fireplace, we need to determine $n(A)$. We are given the number of homes with either a three-car garage or a fireplace, which is $n(A \cup B)$. We are also given the number of homes with a fireplace, $n(B)$, and the number of homes that have both a three-car garage and a fireplace, $n(A \cap B)$. We can use the formula $n(A \cup B) = n(A) + n(B) - n(A \cap B)$ to solve for $n(A)$.

$$\begin{aligned} n(A \cup B) &= n(A) + n(B) - n(A \cap B) \\ 39 &= n(A) + 31 - 18 \\ 39 &= n(A) + 13 \\ 39 - 13 &= n(A) + 13 - 13 \\ 26 &= n(A) \end{aligned}$$

Thus, the number of homes listed for sale that have a three-car garage is 26. ■

Two other set operations are the difference of two sets and the Cartesian product. We will first discuss the difference of two sets.

Difference of Two Sets

Definition: Difference of Two Sets
The **difference of two sets** A and B, symbolized $A - B$, is the set of elements that belong to set A but not to set B.

Using set-builder notation, the difference of two sets A and B is indicated by $A - B = \{x \mid x \in A \text{ and } x \notin B\}$. The shaded region, region I, in Fig. 2.11 represents the difference of two sets A and B, or $A - B$.

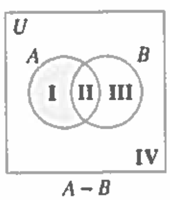


Figure 2.11

Example 9 *The Difference of Two Sets*

Given

$$\begin{aligned} U &= \{a, b, c, d, e, f, g, h, i, j, k\} \\ A &= \{b, d, e, f, g, h\} \\ B &= \{a, b, d, h, i\} \\ C &= \{b, e, g\} \end{aligned}$$

determine

a) $A - B$ b) $A - C$ c) $A' - B$ d) $A - C'$

Solution

a) $A - B$ is the set of elements that are in set A but not set B. The elements that are in set A but not set B are $e, f,$ and g. Therefore, $A - B = \{e, f, g\}$.
b) $A - C$ is the set of elements that are in set A but not set C. The elements that are in set A but not set C are $d, f,$ and h. Therefore, $A - C = \{d, f, h\}$.
c) To determine $A' - B$, we must first determine A'.

$$A' = \{a, c, i, j, k\}$$

$A' - B$ is the set of elements that are in set A' but not set B. The elements that are in set A' but not set B are $c, j,$ and k. Therefore, $A' - B = \{c, j, k\}$.

d) To determine $A - C'$, we must first determine C'.

$$C' = \{a, c, d, f, h, i, j, k\}$$

$A - C'$ is the set of elements that are in set A but not set C'. The elements that are in set A but not set C' are b, e, and g. Therefore, $A - C' = \{b, e, g\}$. ■

Next we discuss the Cartesian product.

Cartesian Product

Definition: Cartesian Product

The **Cartesian product** of set A and set B, symbolized by $A \times B$ and read "A cross B," is the set of all possible *ordered pairs* of the form (a, b), where $a \in A$ and $b \in B$.

To determine the ordered pairs in a Cartesian product, select the first element of set A and form an ordered pair with each element of set B. Then select the second element of set A and form an ordered pair with each element of set B. Continue in this manner until you have used each element of set A.

Example 10 *The Cartesian Product of Two Sets*

Given $A = \{\text{orange, banana, apple}\}$ and $B = \{1, 2\}$, determine the following.

a) $A \times B$ b) $B \times A$ c) $A \times A$ d) $B \times B$

Solution

a) $A \times B = \{$(orange, 1), (orange, 2) (banana, 1), (banana, 2), (apple, 1), (apple, 2)$\}$

b) $B \times A = \{$(1, orange), (1, banana), (1, apple), (2, orange), (2, banana), (2, apple)$\}$

c) $A \times A = \{$(orange, orange), (orange, banana), (orange, apple), (banana, orange), (banana, banana), (banana, apple), (apple, orange), (apple, banana), (apple, apple)$\}$

d) $B \times B = \{(1, 1), (1, 2), (2, 1), (2, 2)\}$ ■

We can see from Example 10 that, in general, $A \times B \neq B \times A$. The ordered pairs in $A \times B$ are not the same as the ordered pairs in $B \times A$ because (orange, 1) $\neq$ (1, orange).

In general, if a set A has m elements and a set B has n elements, then the number of ordered pairs in $A \times B$ will be $m \times n$. In Example 10, set A contains 3 elements and set B contains 2 elements. Notice that $A \times B$ contains 3×2 or 6 ordered pairs.

SECTION 2.3 Exercises

Warm Up Exercises

In exercises 1–8, fill in the blank with an appropriate word, phrase, or symbol(s).

1. The set of all the elements in the universal set that are not in set A is called the ________ of set A.

2. The set containing all the elements that are members of set A or of set B or of both sets is called the ________ of set A and set B.

3. The set containing all the elements that are common to both set A and set B is called the ________ of set A and set B.

4. The set of elements that belong to set A, but not to set B, is called the ________ of two sets A and B.

5. The set of all possible ordered pairs of the form (a, b), where $a \in A$ and $b \in B$, is called the ________ product of set A and set B.

6. If set A has m elements and set B has n elements, the Cartesian product $A \times B$ has ________ elements.

7. Two sets with no elements in common are called ________ sets.

8. In a Venn diagram with two overlapping sets there are ________ regions.

Practice the Skills

In Exercises 9–13, use Fig. 2.2 as a guide to draw a Venn diagram that illustrates the situation described.

9. Set A and set B are disjoint sets.

10. $A \subset B$

11. $B \subset A$

12. $A = B$

13. Set A and set B are overlapping sets.

14. Which set operation is the word *or* generally interpreted to mean?

15. Which set operation is the word *and* generally interpreted to mean?

16. Give the relationship between $n(A \cup B)$, $n(A)$, $n(B)$, and $n(A \cap B)$.

Problem Solving

17. ***Cellular Telephones*** For the sets U, A, and B, construct a Venn diagram and place the elements in the proper regions.

U = {iPhone, Blackberry, LG, DROID, Samsung, Nokia, Motorola, Sony}

A = {iPhone, Blackberry, LG, Motorola, DROID}

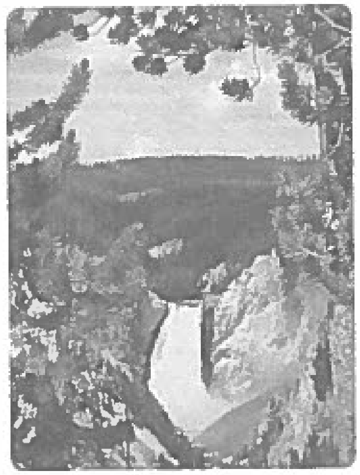
B = {LG, DROID, Nokia, Motorola}

18. ***National Parks*** For the sets U, A, and B, construct a Venn diagram and place the elements in the proper regions.

U = {Badlands, Death Valley, Glacier, Grand Teton, Mammoth Cave, Mount Rainier, North Cascades, Shenandoah, Yellowstone, Yosemite}

A = {Badlands, Glacier, Grand Teton, Mount Rainier, Yellowstone}

B = {Death Valley, Glacier, Mammoth Cave, Mount Rainier, Yosemite}



▲ *Yellowstone National Park*

19. ***Occupations*** The following table shows the fastest-growing occupations for college graduates, based on employment in 2008 and the estimated employment in 2016. Let the occupations in the table represent the universal set.

Fastest-Growing Occupations for College Graduates, 2008–2016		
	Employment (in thousands of jobs)	
Occupation	**2008**	**2016**
Biomedical engineers (BE)	16	28
Network systems analysts (NSA)	292	448
Financial examiners (FE)	27	38
Medical scientists (MS)	109	154
Physicians assistants (PA)	75	104
Biochemists (B)	23	32
Athletic trainers (AT)	16	22
Dental hygienists (DH)	174	237
Veterinary technicians (VT)	80	108
Computer software engineers (CSE)	515	690

Source: U.S. Bureau of Labor Statistics

Let A = the set of fastest-growing occupations for college graduates whose 2008 employment was at least 80,000.

Let B = the set of fastest-growing occupations for college graduates whose estimated employment in 2016 is at least 200,000.

Using the abbreviations listed in the table for each occupation, construct a Venn diagram illustrating the sets.

20. ***Racing Standings*** The following table shows the 2009 NASCAR Sprint Cup Series Final Standings, with the 10 drivers having the highest point total and the number of races won. Let the drivers in the table represent the universal set.

2009 NASCAR Sprint Cup Series Final Standings		
Driver	**Points**	**Wins**
Jimmie Johnson	6652	7
Mark Martin	6511	5
Jeff Gordon	6473	1
Kurt Busch	6446	2
Denny Hamlin	6335	4
Tony Stewart	6309	4
Greg Biffle	6292	0
Juan Montoya	6252	0
Ryan Newman	6175	0
Kasey Kahne	6128	2

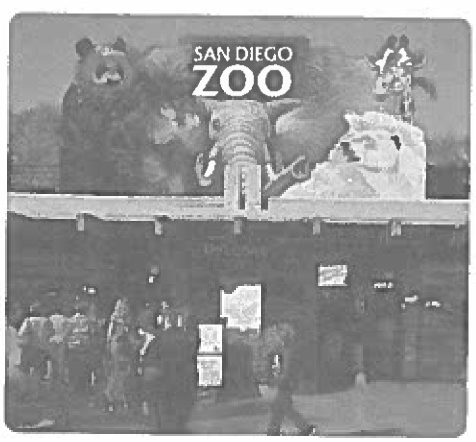
Source: NASCAR

Let A = the set of drivers with more than 6400 points

Let B = the set of drivers with more than 1 win

Construct a Venn diagram illustrating the sets. Use the driver's initials in the Venn diagram.

21. Let U represent the set of animals in U.S. zoos. Let A represent the set of animals in the San Diego zoo. Describe A'.



▲ *San Diego Zoo*

22. Let U represent the set of U.S. colleges and universities. Let A represent the set of U.S. colleges and universities in the state of Mississippi. Describe A'.

In Exercises 23–28,

U is the set of farms in the United States.

A is the set of farms that produce corn.

B is the set of farms that produce tomatoes.

Describe each of the following sets in words.

23. A'

24. B'

25. $A \cup B$

26. $A \cap B$

27. $A \cap B'$

28. $A \cup B'$

In Exercises 29–34,

U is the set of furniture stores.

A is the set of furniture stores that sell mattresses.

B is the set of furniture stores that sell outdoor furniture.

C is the set of furniture stores that sell leather furniture.

Describe the following sets.

29. $A \cup C$

30. $A \cap B$

31. $B' \cap C$

32. $A \cap B \cap C$

33. $A \cup B \cup C$

34. $A' \cup C'$

In Exercises 35–42, use the Venn diagram in Fig. 2.12 to list the set of elements in roster form.

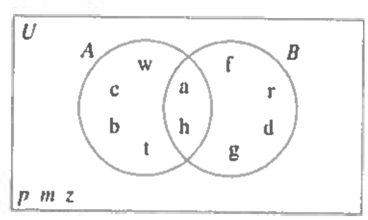


Figure 2.12

35. A

36. B

37. $A \cap B$

38. U

39. $A \cup B$

40. $(A \cup B)'$

41. $A' \cap B'$

42. $(A \cap B)'$

In Exercises 43–50, use the Venn diagram in Fig. 2.13 to list the set of elements in roster form.

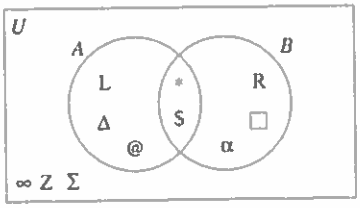


Figure 2.13

43. A

44. B

45. U

46. $A \cap B$

47. $A' \cup B$

48. $A \cup B'$

49. $A' \cap B$

50. $(A \cup B)'$

In Exercises 51–60, let

$$U = \{1, 2, 3, 4, 5, 6, 7, 8\}$$
$$A = \{1, 2, 4, 5, 7\}$$
$$B = \{2, 3, 5, 6\}$$

Determine the following.

51. $A \cup B$

52. $A \cap B$

53. B'

54. $A \cup B'$

55. $(A \cup B)'$

56. $A' \cap B'$

57. $(A \cup B)' \cap B$

58. $(A \cup B) \cap (A \cup B)'$

59. $(B \cup A)' \cap (B' \cup A')$

60. $A' \cup (A \cap B)$

In Exercises 61–70, let

$$U = \{a, b, c, d, e, f, g, h, i, j, k\}$$
$$A = \{a, c, d, f, g, i\}$$
$$B = \{b, c, d, f, g\}$$
$$C = \{a, b, f, i, j\}$$

Determine the following.

61. B'

62. $B \cup C$

63. $A \cap C$

64. $A' \cup B'$

65. $(A \cap C)'$

66. $(A \cap B) \cup C$

67. $A \cup (C \cap B)'$

68. $A \cup (C' \cup B')$

69. $(A' \cup C) \cup (A \cap B)$

70. $(C \cap B) \cap (A' \cap B)$

In Exercises 71–78, let

$$U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$$
$$A = \{1, 2, 4, 6, 9\}$$
$$B = \{1, 3, 4, 5, 8\}$$
$$C = \{4, 5, 9\}$$

Determine the following.

71. $A - B$

72. $A - C$

73. $A - B'$

74. $A' - C$

75. $(A - B)'$

76. $(A - B)' - C$

77. $C - A'$

78. $(C - A)' - B$

In Exercises 79–84, let

$$A = \{a, b, c\}$$
$$B = \{1, 2\}$$

79. Determine $A \times B$.

80. Determine $B \times A$.

81. Does $A \times B = B \times A$?

82. Determine $n(A \times B)$.

83. Determine $n(B \times A)$.

84. Does $n(A \times B) = n(B \times A)$?

Problem Solving

In Exercises 85–98, let

$$U = \{x \mid x \in N \text{ and } x < 10\}$$
$$A = \{x \mid x \in N \text{ and } x \text{ is odd and } x < 10\}$$
$$B = \{x \mid x \in N \text{ and } x \text{ is even and } x < 10\}$$
$$C = \{x \mid x \in N \text{ and } x < 6\}$$

Determine the following.

85. $A \cap B$

86. $A \cup B$

87. $A' \cup B$

88. $(B \cup C)'$

89. $A \cap C'$

90. $A \cap B'$

91. $(B \cap C)'$

92. $(A \cup C) \cap B$

93. $(C' \cup A) \cap B$

94. $(C \cap B) \cup A$

95. $(A \cap B)' \cup C$

96. $(A' \cup C) \cap B$

97. $(A' \cup B') \cap C$

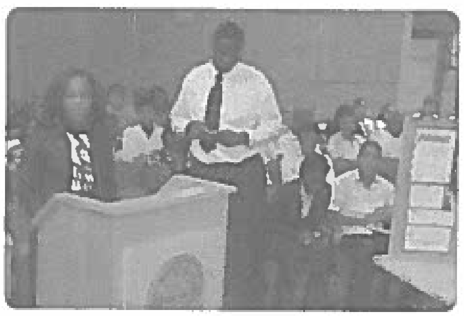
98. $(A' \cap C) \cup (A \cap B)$

99. When will a set and its complement be disjoint? Explain and give an example.

100. When will $n(A \cap B) = 0$? Explain and give an example.

101. ***Pet Ownership*** The results of a survey of customers at PetSmart showed that 27 owned dogs, 38 owned cats, and 16 owned both dogs and cats. How many people owned either a dog or a cat?

102. ***Student Council and Intramurals*** At Madison High School, 46 students participated in student council or intramurals, 30 participated in student council, and 4 participated in student council and intramurals. How many students participated in intramurals?



103. Consider the formula

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

a) Show that this relation holds for $A = \{a, b, c, d\}$ and $B = \{b, d, e, f, g, h\}$.

b) Make up your own sets A and B, each consisting of at least six elements. Using these sets, show that the relation holds.

c) Use a Venn diagram and explain why the relation holds for any two sets A and B.

104. The Venn diagram in Fig. 2.14 shows a technique of labeling the regions to indicate membership of elements in a particular region. Define each of the four regions with a set statement. (*Hint:* $A \cap B'$ defines region I.)

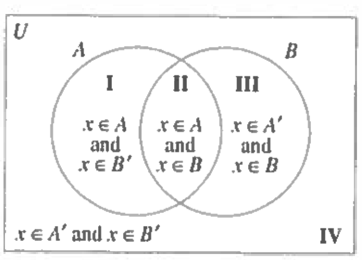


Figure 2.14

In Exercises 105–114, let $U = \{0, 1, 2, 3, 4, 5, \dots\}$, $A = \{1, 2, 3, 4, \dots\}$, $B = \{4, 8, 12, 16, \dots\}$, and $C = \{2, 4, 6, 8, \dots\}$. Determine the following.

105. $A \cup B$

106. $A \cap B$

107. $B \cup C$

108. $B \cap C$

109. $A \cap C$

110. $A' \cap C$

111. $B' \cap C$

112. $(B \cup C)' \cup C$

113. $(A \cap C) \cap B'$

114. $U' \cap (A \cup B)$

Challenge Problems/Group Activities

In Exercises 115–122, determine whether the answer is $\varnothing$, A, or U. (Assume $A \neq \varnothing$, $A \neq U$.)

115. $A \cap A'$

116. $A \cup A'$

117. $A \cup \varnothing$

118. $A \cap \varnothing$

119. $A' \cup U$

120. $A \cap U$

121. $A \cup U$

122. $A \cap A$

In Exercises 123–128, determine the relationship between set A and set B if

123. $A \cap B = B$.

124. $A \cup B = B$.

125. $A \cap B = \varnothing$.

126. $A \cup B = A$.

127. $A \cap B = A$.

128. $A \cup B = \varnothing$.

SECTION 2.4 Venn Diagrams with Three Sets and Verification of Equality of Sets

▲ *Venn diagrams can be used to answer questions regarding the number of people participating in an intramural activity at a college.*

Suppose a college offers intramurals in basketball, flag football, and softball. Is there a way the director of intramurals can determine which students participate in all three activities or which students participate in exactly two of these activities? The answer is yes by using a *Venn Diagram*. A Venn diagram that can be used to convey important information quickly and efficiently.

Why *This is Important* Classifying sets using diagrams often helps us understand the relationship among various sets.

In Section 2.3, we learned how to use Venn diagrams to illustrate two sets. Venn diagrams can also be used to illustrate three sets.

For three sets, A, B, and C, the diagram is drawn so the three sets overlap (Fig. 2.15), creating eight regions. The diagrams in Fig. 2.16 emphasize selected regions of three intersecting sets. *When constructing Venn diagrams with three sets, we generally start with region V and work outward,* as explained in the procedure given on page 69.

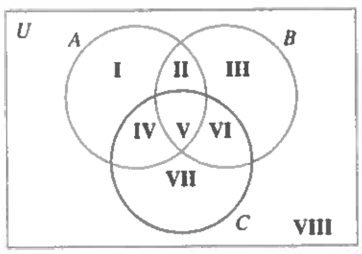


Figure 2.15

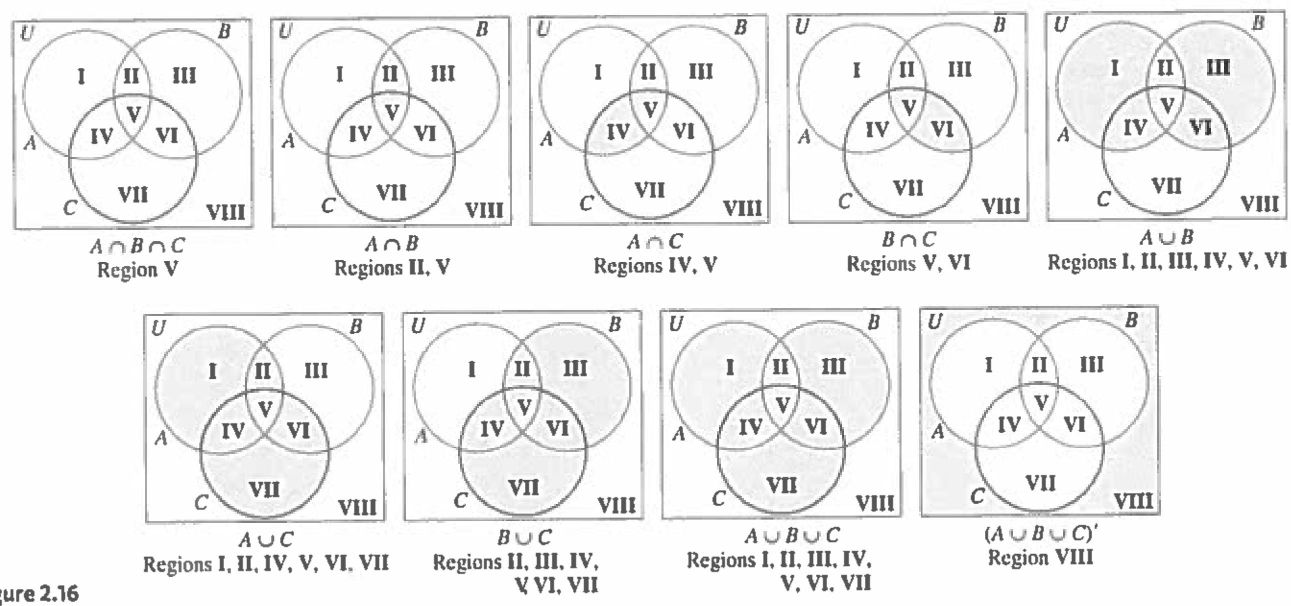


Figure 2.16

PROCEDURE **GENERAL PROCEDURE FOR CONSTRUCTING VENN DIAGRAMS WITH THREE SETS, *A*, *B*, AND *C***

1. Determine the elements to be placed in region V by finding the elements that are common to all three sets, $A \cap B \cap C$.
2. Determine the elements to be placed in region II. Find the elements in $A \cap B$. The elements in this set belong in regions II and V. Place the elements in the set $A \cap B$ that are not listed in region V in region II. The elements in regions IV and VI are found in a similar manner.
3. Determine the elements to be placed in region I by determining the elements in set A that are not in regions II, IV, and V. The elements in regions III and VII are found in a similar manner.
4. Determine the elements to be placed in region VIII by finding the elements in the universal set that are not in regions I through VII.

Example 1 illustrates the general procedure.

Example 1 *Constructing a Venn Diagram for Three Sets*

Construct a Venn diagram illustrating the following sets.

$$U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15\}$$
$$A = \{1, 4, 8, 9, 10, 12\}$$
$$B = \{2, 4, 5, 9, 10, 13\}$$
$$C = \{1, 3, 4, 8, 9, 11\}$$

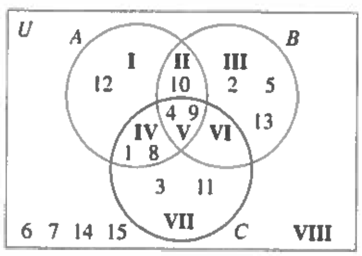


Figure 2.17

Solution First find the intersection of all three sets. Because the elements 4 and 9 are in all three sets, $A \cap B \cap C = \{4, 9\}$. The elements 4 and 9 are placed in region V in Fig.2.17. Next complete region II by determining the intersection of sets A and B.

$$A \cap B = \{4, 9, 10\}$$

$A \cap B$ consists of regions II and V. The elements 4 and 9 have already been placed in region V, so 10 must be placed in region II.

Now determine what numbers go in region IV.

$$A \cap C = \{1, 4, 8, 9\}$$

Since 4 and 9 have already been placed in region V, place the 1 and 8 in region IV. Now determine the numbers to go in region VI.

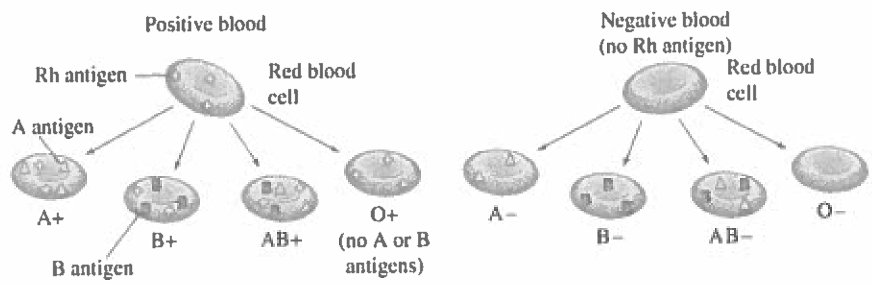
$$B \cap C = \{4, 9\}$$

Since both the 4 and 9 have been placed in region V, there are no numbers to be placed in region VI. Now complete set A. The only element of set A that has not previously been placed in regions II, IV, or V is 12. Therefore, place the element 12 in region I. The element 12 that is placed in region I is only in set A and not in set B or set C. Using set B, complete region III using the same general procedure used to determine the numbers in region I. Using set C, complete region VII by using the same procedure used to complete regions I and III. To determine the elements in region VIII, find the elements in U that have not been placed in regions I–VII. The elements 6, 7, 14, and 15 have not been placed in regions I–VII, so place them in region VIII. ■

Venn diagrams can be used to illustrate and analyze many everyday problems. One example follows.

Example 2 *Blood Types*

Human blood is classified (typed) according to the presence or absence of the specific antigens A, B, and Rh in the red blood cells. Antigens are highly specified proteins and carbohydrates that will trigger the production of antibodies in the blood to fight infection. Blood containing the Rh antigen is labeled positive, +, while blood lacking the Rh antigen is labeled negative, −. Blood lacking both A and B antigens is called type O. Sketch a Venn diagram with three sets A, B, and Rh and place each type of blood listed in the proper region. A person has only one type of blood.



Solution

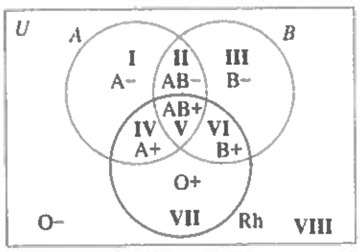


Figure 2.18

As illustrated in Chapter 1, the first thing to do is to read the question carefully and make sure you understand what is given and what you are asked to find. There are three antigens A, B, and Rh. Therefore, begin by naming the three circles in a Venn diagram with the three antigens; see Fig. 2.18.

Any blood containing the Rh antigen is positive, and any blood not containing the Rh antigen is negative. Therefore, all blood in the Rh circle is positive, and all blood outside the Rh circle is negative. The intersection of all three sets, region V, is AB+. Region II contains only antigens A and B and is therefore AB−. Region I is A− because it contains only antigen A. Region III is B−, region IV is A+, and region VI is B+. Region VII is O+, containing only the Rh antigen. Region VIII, which lacks all three antigens, is O−. ■

Verification of Equality of Sets

In this chapter, for clarity we may refer to operations on sets, such as $A \cup B'$ or $A \cap B \cap C$, as *statements involving sets* or simply as *statements*. Now we discuss how to determine if two statements involving sets are equal.

Consider the question: Is $A' \cup B = A' \cap B$ *for all sets A and B*? For the specific sets $U = \{1, 2, 3, 4, 5\}$, $A = \{1, 3\}$, and $B = \{2, 4, 5\}$, is $A' \cup B = A' \cap B$? To answer the question, we do the following.

Find $A' \cup B$	Find $A' \cap B$
$A' = \{2, 4, 5\}$	$A' = \{2, 4, 5\}$
$B = \{2, 4, 5\}$	$B = \{2, 4, 5\}$
$A' \cup B = \{2, 4, 5\}$	$A' \cap B = \{2, 4, 5\}$

For these sets, $A' \cup B = A' \cap B$, because both set statements are equal to $\{2, 4, 5\}$. At this point you may believe that $A' \cup B = A' \cap B$ for all sets A and B.

If we select the sets $U = \{1, 2, 3, 4, 5\}$, $A = \{1, 3, 5\}$, and $B = \{2, 3\}$, we see that $A' \cup B = \{2, 3, 4\}$ and $A' \cap B = \{2\}$. For this case, $A' \cup B \neq A' \cap B$.

Thus, we have proved that $A' \cup B \neq A' \cap B$ for all sets A and B by using a *counterexample.* A counterexample, as explained in Chapter 1, is an example that shows a statement is not true.

In Chapter 1, we explained that proofs involve the use of *deductive reasoning.* Recall that deductive reasoning begins with a general statement and works to a specific conclusion. To verify, or determine whether set statements are equal for any two sets selected, we use deductive reasoning with Venn diagrams. Venn diagrams are used because they can illustrate general cases. To determine if statements that contain sets, such as $(A \cup B)'$ and $A' \cap B'$, are equal for all sets A and B, we use the regions of Venn diagrams. If both statements represent the same regions of the Venn diagram, then the statements are equal for all sets A and B. See Example 3.

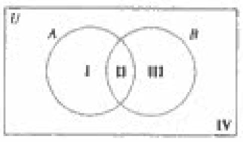


Figure 2.19

Example 3 *Equality of Sets*

Determine whether $(A \cup B)' = A' \cap B'$ for all sets A and B.

Solution Draw a Venn diagram with two sets A and B, as in Fig. 2.19. Label the regions as indicated.

Find $(A \cup B)'$

Set	Corresponding Regions
A	I, II
B	II, III
$A \cup B$	I, II, III
$(A \cup B)'$	IV

Find $A' \cap B'$

Set	Corresponding Regions
A'	III, IV
B'	I, IV
$A' \cap B'$	IV

Both statements are represented by the same region, IV, of the Venn diagram. Thus, $(A \cup B)' = A' \cap B'$ for all sets A and B. ■

In Example 3, when we proved that $(A \cup B)' = A' \cap B'$, we started with two general sets and worked to the specific conclusion that both statements represented the same regions of the Venn diagram. We showed that $(A \cup B)' = A' \cap B'$ *for all sets A and B.* No matter what sets we choose for A and B, this statement will be true. For example, let $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, $A = \{3, 4, 6, 10\}$, and $B = \{1, 2, 4, 5, 6, 8\}$.

$$\begin{aligned} (A \cup B)' &= A' \cap B' \\ \{1, 2, 3, 4, 5, 6, 8, 10\}' &= \{3, 4, 6, 10\}' \cap \{1, 2, 4, 5, 6, 8\}' \\ \{7, 9\} &= \{1, 2, 5, 7, 8, 9\} \cap \{3, 7, 9, 10\} \\ \{7, 9\} &= \{7, 9\} \end{aligned}$$

We can also use Venn diagrams to prove statements involving three sets.

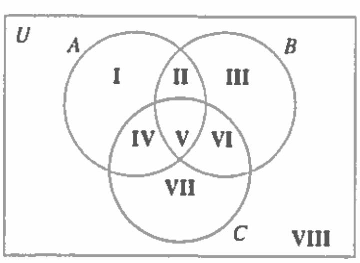


Figure 2.20

Example 4 *Equality of Sets*

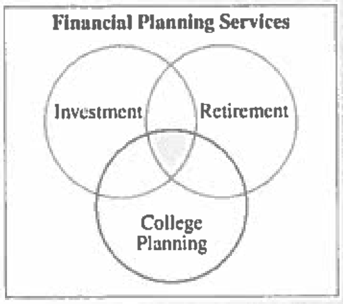
Determine whether $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ for all sets, A, B, and C.

Solution Because the statements include three sets, A, B, and C, three circles must be used. The Venn diagram illustrating the eight regions is shown in Fig. 2.20.

First we will find the regions that correspond to $A \cup (B \cap C)$, and then we will find the regions that correspond to $(A \cup B) \cap (A \cup C)$. If both answers are the same, the statements are equal.

MATHEMATICS TODAY

Using Venn Diagrams



A financial planning company uses the Venn diagram above to illustrate the financial planning services it offers. From the diagram, we can see that this company offers advice in an "intersection" of the areas investment, retirement, and college planning, the intersection of all three sets.

We categorize items on a daily basis. Children are taught how to categorize items at an early age when they learn how to classify items according to color, shape, and size. Biologists categorize items when they classify organisms according to shared characteristics.

Why This is Important A Venn diagram is a very useful tool to help order and arrange items and to picture the relationship between sets.

Find $A \cup (B \cap C)$

Set	Corresponding Regions
A	I, II, IV, V
$B \cap C$	V, VI
$A \cup (B \cap C)$	I, II, IV, V, VI

Find $(A \cup B) \cap (A \cup C)$

Set	Corresponding Regions
$A \cup B$	I, II, III, IV, V, VI
$A \cup C$	I, II, IV, V, VI, VII
$(A \cup B) \cap (A \cup C)$	I, II, IV, V, VI

The regions that correspond to $A \cup (B \cap C)$ are I, II, IV, V, and VI, and the regions that correspond to $(A \cup B) \cap (A \cup C)$ are also I, II, IV, V, and VI. The results show that both statements are represented by the same regions, namely, I, II, IV, V, and VI, and therefore $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ for all sets A, B, and C. ■

In Example 4, we proved that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ for all sets A, B, and C. Show that this statement is true for the specific sets $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, $A = \{1, 2, 3, 7\}$, $B = \{2, 3, 4, 5, 7, 9\}$, and $C = \{1, 4, 7, 8, 10\}$.

De Morgan's Laws

In set theory, logic, and other branches of mathematics, a pair of related theorems known as De Morgan's laws make it possible to transform statements and formulas into alternative and often more convenient forms. In set theory, ***De Morgan's laws*** are symbolized as follows.

De Morgan's Laws

1. $(A \cap B)' = A' \cup B'$
2. $(A \cup B)' = A' \cap B'$

Law 2 was verified in Example 3. We suggest that you verify law 1 at this time. The laws were expressed verbally by William of Ockham in the fourteenth century. In the nineteenth century, Augustus De Morgan expressed them mathematically. De Morgan's laws will be discussed more thoroughly in Chapter 3, Logic.

SECTION 2.4 Exercises

Warm Up Exercises

In Exercises 1–4, fill in the blank with an appropriate word, phrase, or symbol(s).

1. The number of regions created when constructing a Venn diagram with three overlapping sets is ________ .

2. a) When constructing a Venn diagram with three overlapping sets, region ________ is generally completed first.

b) When constructing a Venn diagram with three overlapping sets, after completing region V, the next regions generally completed are II, IV, and ________ .

3. Complete DeMorgan's laws:

a) $(A \cup B)' =$ ________

b) $(A \cap B)' =$ ________

4. When using Venn diagrams to verify or determine whether set statements are equal we use ________ reasoning.

Practice the Skills/Problem Solving

5. A Venn diagram contains three sets, A, B, and C, as in Fig. 2.15 on page 68. If region V contains 4 elements and there are 12 elements in $B \cap C$, how many elements belong in region VI? Explain.

6. A Venn diagram contains three sets, A, B, and C, as in Fig. 2.15 on page 68. If region V contains 4 elements and there are 9 elements in $A \cap B$, how many elements belong in region II? Explain.

7. a) For $U = \{1, 2, 3, 4, 5\}$, $A = \{1, 4, 5\}$, and $B = \{1, 4, 5\}$, does $A \cup B = A \cap B$?

b) By observing the answer to part (a), can we conclude that $A \cup B = A \cap B$ for all sets A and B? Explain.

c) Using a Venn diagram, determine if $A \cup B = A \cap B$ for all sets A and B.

8. Construct a Venn diagram illustrating the following sets.

$U = \{a, b, c, d, e, f, g, h, i, j\}$
$A = \{c, d, e, g, h, i\}$
$B = \{a, c, d, g\}$
$C = \{c, f, i, j\}$

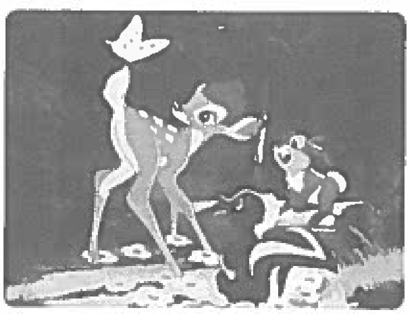
9. Construct a Venn diagram illustrating the following sets.

U = {*Cinderella, Pinocchio, Ratatouille, Fantasia, Dumbo, Bambi, Pocahontas, Hercules, Mulan, Tarzan, Cars*}

A = {*Bambi, Hercules, Pocahontas, Tarzan*}

B = {*Ratatouille, Bambi, Mulan, Hercules*}

C = {*Pocahontas, Cinderella, Bambi, Ratatouille, Fantasia*}



▲ *Bambi*

10. Construct a Venn diagram illustrating the following sets.

U = {microwave oven, freezer, dishwasher, refrigerator, washer, dryer, toaster, blender, food processor, iron}

A = {toaster, blender, iron, dishwasher, washer, dryer}

B = {dishwasher, iron, freezer}

C = {washer, dryer, iron, freezer, microwave oven}

11. Construct a Venn diagram illustrating the following sets.

U = {American Eagle, Best Buy, Wal-Mart, Kmart, Target, Sears, JCPenney, Costco, Kohl's, Gap, Gap Kids, Foot Locker, Old Navy, Macy's}

A = {American Eagle, Wal-Mart, Target, JCPenney, Old Navy}

B = {Best Buy, Target, Costco, Old Navy, Macy's}

C = {Target, Sears, Kohl's, Gap, JCPenney}

12. Construct a Venn diagram illustrating the following sets.

U = {Louis Armstrong, Glenn Miller, Stan Kenton, Charlie Parker, Duke Ellington, Benny Goodman, Count Basie, John Coltrane, Dizzy Gillespie, Miles Davis, Thelonius Monk}

A = {Stan Kenton, Count Basie, Dizzy Gillespie, Duke Ellington, Thelonius Monk}

B = {Louis Armstrong, Glenn Miller, Count Basie, Duke Ellington, Miles Davis}

C = {Count Basie, Miles Davis, Stan Kenton, Charlie Parker, Duke Ellington}

13. ***Olympic Medals*** Consider the following table, which shows countries that won at least 25 medals in the 2008 Summer Olympics. Let the countries in the table represent the universal set.

Country	Gold Medals	Silver Medals	Bronze Medals	Total Medals
United States	36	38	36	110
China	51	21	28	100
Russia	23	21	28	72
Great Britain	19	13	15	47
Australia	14	15	17	46
Germany	16	10	15	41
France	7	16	17	40
South Korea	13	10	8	31
Italy	8	10	10	28
Ukraine	7	5	15	27
Japan	9	6	10	25

Source: United States Olympic Committee.

Let A = set of countries that won at least 30 gold medals.
Let B = set of countries that won at least 15 silver medals.
Let C = set of countries that won at least 10 bronze medals.

Construct a Venn diagram that illustrates the sets A, B, and C.

14. ***Popular TV Shows*** Construct a Venn diagram illustrating the following sets.

U = {*American Idol (AI), CSI, Dancing with the Stars (DWS), Family Guy (FG), Gossip Girl (GG), Monday Night Football (MNF), NCIS, Sunday Night Football (SNF), Survivor (S)*}.

A = {*AI, CSI, DWS, SNF, NCIS*}

B = {*AI, DWS, SNF, NCIS, MNF*}

C = {*AI, CSI, SNF, NCIS, MNF, S*}

Rankings of Fruit-Producing Countries For Exercises 15–20, use the following table, which shows the top 10 countries for production of apples, oranges, and nuts. The universal set is the set of countries listed in the world.

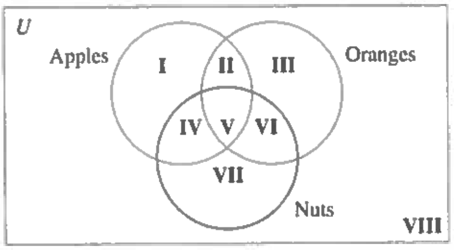


Figure 2.21

Ranking of Fruit-Producing countries		
Apples	**Oranges**	**Nuts**
1. China	1. Brazil	1. United States
2. United States	2. United States	2. Indonesia
3. Iran	3. Mexico	3. Mexico
4. Turkey	4. India	4. Ethiopia
5. Russia	5. China	5. China
6. Italy	6. Spain	6. Australia
7. India	7. Indonesia	7. Guatemala
8. France	8. Iran	8. Portugal
9. Chile	9. Italy	9. Thailand
10. Argentina	10. Egypt	10. Philippines

Source: Food and Agriculture of the United Nations

Indicate in which region, I–VIII in Fig. 2.21, each of the following countries belongs.

15. Italy

16. United States

17. Canada

18. Portugal

19. Spain

20. Mexico

Figures *In Exercises 21–32, indicate in Fig. 2.22 the region in which each of the figures would be placed.*

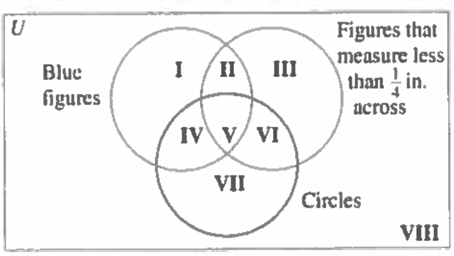


Figure 2.22

21. 

22. 

23. 

24. 

25. 

26. 

27. 

28. 

29. 

30.

31. 

32. 

Senate Bills *During a session of the U.S. Senate, three bills were voted on. The votes of six senators are shown below the figure. Determine in which region of Fig. 2.23 each senator would be placed. The set labeled Bill 1 represents the set of senators who voted yes on Bill 1, and so on.*

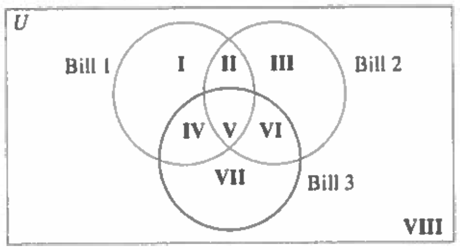


Figure 2.23

SENATOR	BILL 1	BILL 2	BILL 3
33. Hutchinson	yes	no	no
34. Kerry	no	no	yes
35. McCain	no	no	no
36. Mikulski	yes	yes	yes
37. Rand	no	yes	yes
38. Reid	no	yes	no

In Exercises 39–52, use the Venn diagram in Fig. 2.24 to list the sets in roster form.

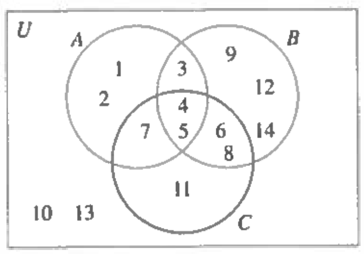


Figure 2.24

39. A

40. U

41. B

42. C

43. $A \cap B$

44. $A \cap C$

45. $(B \cap C)'$

46. $A \cap B \cap C$

47. $A \cup B$

48. $B \cup C$

49. $(A \cup C)'$

50. $A \cap (B \cup C)$

51. A'

52. $(A \cup B \cup C)'$

In Exercises 53–60, use Venn diagrams to determine whether the following statements are equal for all sets A and B.

53. $(A \cap B)'$, $A' \cup B'$

54. $(A \cap B)'$, $A' \cup B$

55. $A' \cup B'$, $A \cap B$

56. $(A \cup B)'$, $(A \cap B)'$

57. $A' \cup B'$, $(A \cup B)'$

58. $A' \cap B'$, $A \cup B'$

59. $(A' \cap B)'$, $A \cup B'$

60. $A' \cap B'$, $(A' \cap B')'$

In Exercises 61–70, use Venn diagrams to determine whether the following statements are equal for all sets A, B, and C.

61. $A \cap (B \cup C)$, $(A \cap B) \cup C$

62. $A \cup (B \cap C)$, $(B \cap C) \cup A$

63. $A \cap (B \cup C)$, $(B \cup C) \cap A$

64. $A \cup (B \cap C)'$, $A' \cap (B' \cup C)$

65. $A \cap (B \cup C)$, $(A \cap B) \cup (A \cap C)$

66. $A \cup (B \cap C)$, $(A \cup B) \cap (A \cup C)$

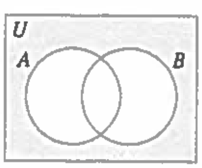
67. $A \cup (B \cup C)'$, $A \cup (B' \cap C')$

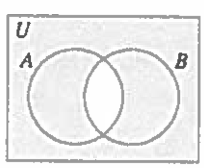
68. $(A \cup B) \cap (B \cup C)$, $B \cup (A \cap C)$

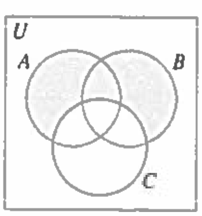
69. $(A \cup B)' \cap C$, $(A' \cup C') \cap (B' \cup C)$

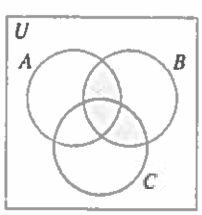
70. $(C \cap B)' \cup (A \cap B)'$, $A \cap (B \cap C)$

In Exercises 71–74, use set statements to write a description of the shaded area. Use union, intersection and complement as necessary. More than one answer may be possible.

71. 

72. 

73. 

74. 

75. Let

$$U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$$
$$A = \{1, 2, 3, 4\}$$
$$B = \{3, 6, 7\}$$
$$C = \{6, 7, 9\}$$

a) Show that $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$ for these sets.

b) Make up your own sets A, B, and C. Verify that $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$ for your sets A, B, and C.

c) Use Venn diagrams to verify that $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$ for all sets A, B, and C.

76. Let

$$U = \{a, b, c, d, e, f, g, h, i\}$$
$$A = \{a, c, d, e, f\}$$
$$B = \{c, d\}$$
$$C = \{a, b, c, d, e\}$$

a) Determine whether $(A \cup C)' \cap B = (A \cap C)' \cap B$ for these sets.

b) Make up your own sets, A, B, and C. Determine whether $(A \cup C)' \cap B = (A \cap C)' \cap B$ for your sets.

c) Determine whether $(A \cup C)' \cap B = (A \cap C)' \cap B$ for all sets A, B, and C.

77. ***Blood Types*** A hematology text gives the following information on percentages of the different types of blood worldwide.

Type	Positive Blood, %	Negative Blood, %
A	37	6
O	32	6.5
B	11	2
AB	5	0.5

Construct a Venn diagram similar to the one in Example 2 and place the correct percentage in each of the eight regions.

78. Define each of the eight regions in Fig. 2.25 using sets A, B, and C and a set operation. (*Hint:* $A \cap B' \cap C'$ defines region I.)

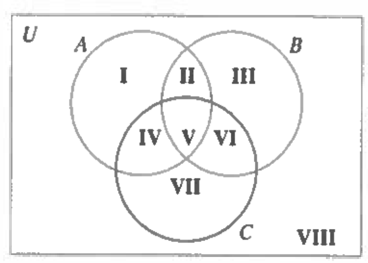


Figure 2.25

79. *Categorizing Contracts* J & C Mechanical Contractors wants to classify its projects. The contractors categorize set A as construction projects, set B as plumbing projects, and set C as projects with a budget greater than \$300,000.

a) Draw a Venn diagram that can be used to categorize the company projects according to the listed criteria.

b) Determine the region of the diagram that contains construction projects and plumbing projects with a budget greater than \$300,000. Describe the region using sets A, B, and C with set operations. Use union, intersection, and complement as necessary.

c) Determine the region of the diagram that contains plumbing projects with a budget greater than \$300,000 that are not construction projects. Describe the region using sets A, B and C with set operations. Use union, intersection, and complement as necessary.

d) Determine the region of the diagram that contains construction projects and nonplumbing projects whose budget is less than or equal to \$300,000. Describe the region using sets A, B, and C with set operations. Use union, intersection, and complement as necessary.

Challenge Problem/Group Activity

80. We were able to determine the number of elements in the union of two sets with the formula

$$n(A \cup B) = n(A) + n(B) - n(A \cap B).$$

Can you determine a formula for finding the number of elements in the union of three sets? In other words, write a formula to determine $n(A \cup B \cup C)$. [*Hint*: The formula will contain each of the following: $n(A)$, $n(B)$, $n(C)$, $n(A \cap B \cap C')$, $n(A \cap B' \cap C)$, $n(A' \cap B \cap C)$, and $2n(A \cap B \cap C)$.]

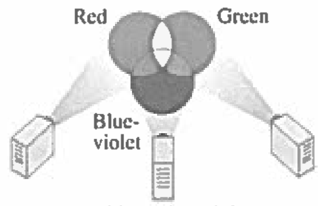
Recreational Mathematics

81. a) Construct a Venn diagram illustrating four sets, A, B, C, and D. (*Hint*: Four circles cannot be used, and you should end up with 16 *distinct* regions.) Have fun!

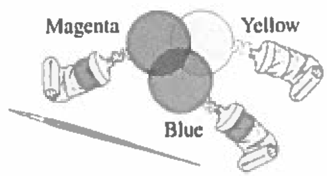
b) Label each region with a set statement (see Exercise 78). Check all 16 regions to make sure that *each is distinct*.

Internet/Research Activity

82. The two Venn diagrams below illustrate what happens when colors are added or subtracted. Do research in an art text, an encyclopedia, the Internet, or another source and write a report explaining the creation of the colors in the Venn diagrams, using such terms as union of colors and subtraction (or difference) of colors.



Additive color mixing



Subtractive color mixing

SECTION 2.5 Applications of Sets


▲ *We can use a Venn diagram to determine how many members of a health club took a particular fitness class.*

The members of a health club were surveyed about taking fitness classes at the club. Suppose the results of the survey show how many members took a yoga class, how many members took a spinning class, and how many members took a class in yoga and a class in spinning. How can the manager of the club use this information to determine how many members took only a yoga class? In this section, we will learn how to use Venn diagrams to answer this type of question.

Why ***This is Important*** As you read through this section, you will see many real-life applications of set theory.

We can solve practical problems involving sets by using the problem-solving process discussed in Chapter 1: Understand the problem, devise a plan, carry out the plan, and then examine and check the results. First determine: What is the problem? or What am I looking for? To devise the plan, list all the facts that are given and how they are related. *Look for key words or phrases* such as "only set A," "set A and set B," "set A or set B," "set A and set B and not set C." Remember that *and* means intersection, *or* means union, and *not* means complement. The problems we solve in this section contain two or three sets of elements, which can be represented in a Venn diagram. Our plan will generally include drawing a Venn diagram, labeling the diagram, and filling in the regions of the diagram.

Whenever possible, follow the procedure in Section 2.4 for completing the Venn diagram and then answer the questions. *Remember: When drawing Venn diagrams, we generally start with the intersection of the sets and work outward.*

Example 1 *Yogurt Taste Test*

A yogurt company wishes to introduce a new yogurt flavor. The company is considering two flavors: raspberry cheesecake (R) and orange creme (O). In a survey of 250 people it was found that

180 people liked raspberry cheesecake.

139 people liked orange creme.

82 people liked both flavors.

Of those surveyed, how many people

a) did not like either raspberry cheesecake or orange creme?
b) liked raspberry cheesecake, but not orange creme?
c) liked orange creme, but not raspberry cheesecake?
d) liked either raspberry cheesecake or orange creme?

Solution The problem provides the following information.

The number of people surveyed is 250: $n(U) = 250$.

The number of people surveyed who liked raspberry cheesecake is 180: $n(R) = 180$.

The number of people surveyed who liked orange creme is 139: $n(O) = 139$.

The number of people surveyed who liked both raspberry cheesecake and orange creme is 82: $n(R \cap O) = 82$.

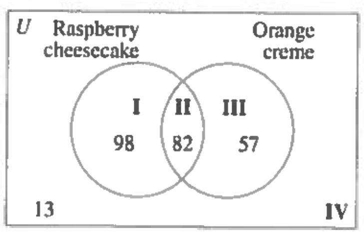


Figure 2.26

We illustrate this information on the Venn diagram shown in Fig. 2.26. We already know that $R \cap O$ corresponds to region II. Because $n(R \cap O) = 82$, we write 82 in region II. Set R consists of regions I and II. We know that set R, the number of people who liked raspberry cheesecake, contains 180 people. Therefore, region I contains $180 - 82 = 98$ people. We write the number 98 in region I. Set O consists of regions II and III. Because $n(O) = 139$, the total in these two regions must be 139. Region II contains 82, leaving $139 - 82$, or 57, for region III. We write 57 in region III.

The total number of people surveyed who liked raspberry cheesecake or orange creme is found by adding the numbers in regions I, II, and III. Therefore $n(R \cup O) = 98 + 82 + 57 = 237$. The number in region IV is the difference between $n(U)$ and $n(R \cup O)$. There are $250 - 237$, or 13, members in region IV.

a) The people surveyed who did not like either raspberry cheesecake or orange creme are those people in the universal set who are not contained in set R or set O. The 13 people in region IV did not like raspberry cheesecake or orange creme.
b) The 98 people in region I are those people surveyed who liked raspberry cheesecake, but not orange creme.
c) The 57 people in region III are those people surveyed who liked orange creme, but not raspberry cheesecake.
d) The people in regions I, II, or III are those people surveyed who liked either raspberry cheesecake or orange creme. Thus, $98 + 82 + 57$, or 237, people surveyed liked either raspberry cheesecake or orange creme. Notice that the 82 people in region II who like both flavors are included in those people surveyed who liked either raspberry cheesecake or orange creme. ■

Similar problems involving three sets can be solved, as illustrated in Example 2.

Example 2 *Software Purchases*

CompUSA has recorded recent sales for three types of computer software: games, educational software, and utility programs. The following information regarding software purchases was obtained from a survey of 893 customers.

545 purchased games.
497 purchased educational software.
290 purchased utility programs.
297 purchased games and educational software.
196 purchased educational software and utility programs.
205 purchased games and utility programs.
157 purchased all three types of software.

Use a Venn diagram to answer the following questions. How many customers purchased

a) none of these types of software?
b) only games?
c) at least one of these types of software?
d) exactly two of these types of software?

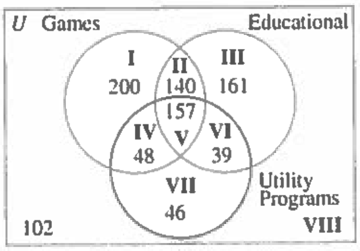


Figure 2.27

Solution Begin by constructing a Venn diagram with three overlapping circles. One circle represents games, another educational software, and the third utilities. See Fig 2.27. Label the eight regions.

Whenever possible, work from the center of the diagram outward. First fill in region V. Since 157 customers purchased all three types of software, we place 157 in region V. Next determine the number to be placed in region II. Regions II and V together represent the customers who purchased both games and educational software. Since 297 customers purchased both of these types of software, the sum of the numbers in these regions must be 297. Since 157 have already been placed in region V, $297 - 157 = 140$ must be placed in region II. Now we determine the number to be placed in region IV. Since 205 customers purchased both games and utility programs, the sum of the numbers in regions IV and V must be 205. Since 157 have already been placed in region V, $205 - 157 = 48$ must be placed in region IV. Now determine the number to be placed in region VI. A total of 196 customers purchased educational software and utility programs. The numbers in regions V and VI must total 196. Since 157 have already been placed in region V, the number to be placed in region VI is $196 - 157 = 39$.

Now that we have determined the numbers for regions V, II, IV, and VI, we can determine the numbers to be placed in regions I, III, and VII. We are given that 545 customers purchased games. The sum of the numbers in regions I, II, IV, and V must be 545. To determine the number to be placed in region I, subtract the amounts in regions II, IV, and V from 545. There must be $545 - 140 - 48 - 157 = 200$ in region I. Determine the numbers to be placed in regions III and VII in a similar manner.

$$\text{Region III} = 497 - 140 - 157 - 39 = 161$$
$$\text{Region VII} = 290 - 48 - 157 - 39 = 46$$

Now that we have determined the numbers in regions I through VII, we can determine the number to be placed in region VIII. Adding the numbers in regions I through VII yields a sum of 791. The difference between the total number of customers surveyed, 893, and the sum of the numbers in regions I through VII must be placed in region VIII.

$$\text{Region VIII} = 893 - 791 = 102$$

Now that we have completed the Venn diagram, we can answer the questions.

a) One hundred two customers did not purchase any of these types of software. These customers are indicated in region VIII.

TIMELY TIP

When constructing a Venn diagram, the most common mistake made by students is forgetting to subtract the number in region V from the respective values in determining the numbers to be placed in regions II, IV, and VI.

b) Region I represents those customers who purchased only games. Thus, 200 customers purchased only games.

c) The words *at least one* mean "one or more." All those in regions I through VII purchased at least one of the types of software. The sum of the numbers in regions I through VII is 791, so 791 customers purchased at least one of the types of software.

d) The customers in regions II, IV, and VI purchased exactly two of the types of software. Summing the numbers in these regions 140 + 48 + 39 we find that 227 customers purchased exactly two of these types of software. Notice that we did not include the customers in region V. Those customers purchased all three types of software. ■

The procedure to work problems like those given in Example 2 is generally the same. Start by completing region V. Next complete regions II, IV, and VI. Then complete regions I, III, and VII. Finally, complete region VIII. When you are constructing Venn diagrams, be sure to check your work carefully.

Example 3 *Travel Packages*

Liberty Travel surveyed 125 potential customers. The following information was obtained.

68 wished to travel to Hawaii.
53 wished to travel to Las Vegas.
47 wished to travel to Disney World.
34 wished to travel to Hawaii and Las Vegas.
26 wished to travel to Las Vegas and Disney World.
23 wished to travel to Hawaii and Disney World.
18 wished to travel to all three destinations.

▲ *Hawaii*

Use a Venn diagram to answer the following questions. How many of those surveyed

a) did not wish to travel to any of these destinations?
b) wished to travel only to Hawaii?
c) wished to travel to Disney World *and* Las Vegas, but not to Hawaii?
d) wished to travel to Disney World *or* Las Vegas, but not to Hawaii?
e) wished to travel to exactly one of these destinations?

Solution The Venn diagram is constructed using the procedures we outlined in Example 2. The diagram is illustrated in Fig. 2.28. We suggest you construct the diagram by yourself now and check your diagram with Fig. 2.28.

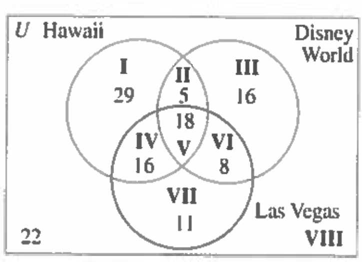


Figure 2.28

a) Twenty-two potential customers did not wish to travel to any of these destinations (see region VIII).

b) Twenty-nine potential customers wished to travel only to Hawaii (see region I).

c) Those potential customers in region VI wished to travel to Disney World *and* Las Vegas, but not to Hawaii. Therefore, eight customers satisfied the criteria.

d) The word *or* in this type of problem means one or the other or both. All the potential customers in regions II, III, IV, V, VI, and VII wished to travel to Disney World or Las Vegas. Those in regions II, IV, and V also wished to travel to Hawaii. The potential customers that wished to travel to Disney World or Las Vegas, but not to Hawaii, are found by adding the numbers in regions III, VI, and VII. There are 16 + 8 + 11 = 35 potential customers who satisfy the criteria.

e) Those potential customers in regions I, III, and VII wished to travel to exactly one of the destinations. Therefore, 29 + 16 + 11 = 56 customers wished to travel to exactly one of these destinations. ■

SECTION 2.5 Exercises

Practice The Skills/Problem Solving

In Exercises 1–15, draw a Venn diagram to obtain the answers.

1. *Market Purchases* During the fall festival at Wambach's Farmer's market, 200 customers made the following purchases.
   109 purchased pumpkins.
   98 purchased pies.
   61 purchased both pumpkins and pies.

   Of those surveyed,

   a) how many purchased only pumpkins?

   b) how many purchased only pies?

   c) how many did not purchase either of these items?

2. *Landscape Purchases* Agway Lawn and Garden collected the following information regarding purchases from 130 of its customers.
   74 purchased shrubs.
   70 purchased trees.
   41 purchased both shrubs and trees.

   Of those surveyed,

   a) how many purchased only shrubs?

   b) how many purchased only trees?

   c) how many did not purchase either of these items?

3. *Real Estate* The Maiellos are moving to Wilmington, Delaware. Their real estate agent located 83 houses listed for sale, in the Wilmington area, in their price range. Of these houses listed for sale,
   47 had a family room.
   42 had a deck.
   30 had a family room and a deck.

   How many had

   a) a family room but not a deck?

   b) a deck but not a family room?

   c) either a family room or a deck?

4. *Racing* Fleet Foot Racing interviewed 150 long-distance runners to determine the type of races in which they participated. The following information was determined.
   102 participated in a marathon.
   93 participated in a triathlon.
   55 participated in both a marathon and a triathlon.

   How many

   a) participated in only a marathon?

   b) participated in only a triathlon?

   c) participated in either a marathon or a triathlon?

   d) had not participated in either a marathon or a triathlon?

5. *Cultural Activities* Thirty-three U.S. cities were researched to determine whether they had a professional sports team, a symphony, or a children's museum. The following information was determined.
   16 had a professional sports team.
   17 had a symphony.
   15 had a children's museum.
   11 had a professional sports team and a symphony.
   7 had a professional sports team and a children's museum.
   9 had a symphony and children's museum.
   5 had all three activities.

   How many of the cities surveyed had

   a) only a professional sports team?

   b) a professional sports team and a symphony, but not a children's museum?

   c) a professional sports team or a symphony?

   d) a professional sports team or a symphony, but not a children's museum?

   e) exactly two of the activities?

6. *Amusement Parks* In a survey of 85 amusement parks, it was found that
   24 had a hotel on site.
   55 had water slides.
   38 had a wave pool.
   13 had a hotel on site and water slides.
   10 had a hotel on site and a wave pool.
   19 had water slides and a wave pool.
   7 had all three features.

How many of the amusement parks surveyed had

a) only water slides?

b) exactly one of these features?

c) at least one of these features?

d) exactly two of these features?

e) none of these features?

7. *Book Purchases* A survey of 85 customers was taken at Barnes & Noble regarding the types of books purchased. The survey found that

44 purchased mysteries.
33 purchased science fiction.
29 purchased romance novels.
13 purchased mysteries and science fiction.
5 purchased science fiction and romance novels.
11 purchased mysteries and romance novels.
2 purchased all three types of books.

How many of the customers surveyed purchased

a) only mysteries?

b) mysteries and science fiction, but not romance novels?

c) mysteries or science fiction?

d) mysteries or science fiction, but not romance novels?

e) exactly two types?

8. *Movies* A survey of 350 customers was taken at Regal Cinemas in Austin, Texas, regarding the type of movies customers liked. The following information was determined.

196 liked dramas.
153 liked comedies.
88 liked science fiction.
59 liked dramas and comedies.
37 liked dramas and science fiction.
32 liked comedies and science fiction.
21 liked all three types of movies.

Of the customers surveyed, how many liked

a) none of these types of movies?

b) only dramas?

c) exactly one of these types of movies?

d) exactly two of these types of movies?

e) dramas or comedies?

9. *Jobs at a Restaurant* Panera Bread compiled the following information regarding 30 of its employees. The following was determined.

8 cooked food.
9 washed dishes.
18 operated the cash register.
4 cooked food and washed dishes.
5 washed dishes and operated the cash register.
3 cooked food and operated the cash register.
2 did all three jobs.

How many of the employees

a) only cooked food ?

b) only operated the cash register?

c) washed dishes and operated the cash register but did not cook food?

d) washed dishes or operated the cash register but did not cook food?

e) did at least two of these jobs?

10. *Electronic Devices* In a survey of college students, it was found that

356 owned an iPod.
293 owned a laptop.
285 owned a gaming system.
193 owned an iPod and a laptop.
200 owned an iPod and a gaming system.
139 owned a laptop and a gaming system.
68 owned an iPod, a laptop, and a gaming system.
26 owned none of these devices.

a) How many college students were surveyed?

Of the college students surveyed, how many owned

b) an iPod and a gaming system, but not a laptop?

c) a laptop, but neither an iPod nor a gaming system?

d) exactly two of these devices?

e) at least one of these devices?

11. *Homeowners' Insurance Policies* A committee of the Florida legislature decided to analyze 350 homeowners' insurance policies to determine if the consumers' homes

were covered for damage due to sinkholes, mold, and floods. The following results were determined.

170 homes were covered for damage due to sinkholes.
172 homes were covered for damage due to mold.
234 homes were covered for damage due to floods.
105 homes were covered for damage due to sinkholes and mold.
115 homes were covered for damage due to mold and floods.
109 homes were covered for damage due to sinkholes and floods.
78 homes were covered for damage due to all three conditions.

How many of the homes

a) were covered for damage due to mold but were not covered for damage due to sinkholes?

b) were covered for damage due to sinkholes or mold?

c) were covered for damage due to mold and floods but were not covered for damage due to sinkholes?

d) were not covered for damage due to any of the three conditions?

12. *Appetizers Survey* Da Tulio's Restaurant hired Dennis Goldstein to determine what kind of appetizers customers liked. He surveyed 100 people, with the following results: 78 liked shrimp cocktail, 56 liked mozzarella sticks, and 35 liked both shrimp cocktail and mozzarella sticks. Every person interviewed liked one or the other or both kinds of appetizers. Does this result seem correct? Explain your answer.

13. *Discovering an Error* An immigration agent sampled cars going from the United States into Canada. In his report, he indicated that of the 85 cars sampled,

35 cars were driven by women.
53 cars were driven by U.S. citizens.
43 cars had two or more passengers.
27 cars were driven by women who are U.S. citizens.
25 cars were driven by women and had two or more passengers.
20 cars were driven by U.S. citizens and had two or more passengers.
15 cars were driven by women who are U.S. citizens and had two or more passengers.

After his supervisor reads the report, she explains to the agent that he made a mistake. Explain how his supervisor knew that the agent's report contained an error.

Challenge Problems/Group Activities

14. *Parks* A survey of 300 parks showed the following.

15 had only camping.
20 had only hiking trails.
35 had only picnicking.
185 had camping.
140 had camping and hiking trails.
125 had camping and picnicking.
210 had hiking trails.

Determine the number of parks that

a) had at least one of these features.

b) had all three features.

c) did not have any of these features.

d) had exactly two of these features.

15. *Surveying Farmers* A survey of 500 farmers in a midwestern state showed the following.

125 grew only wheat.
110 grew only corn.
90 grew only oats.
200 grew wheat.
60 grew wheat and corn.
50 grew wheat and oats.
180 grew corn.

Determine the number of farmers who

a) grew at least one of the three.

b) grew all three.

c) did not grow any of the three.

d) grew exactly two of the three.

16. ***Family Reunion*** When the Montesano family members discussed where their annual reunion should take place, they found that of all the family members,

8 would not go to a park.
7 would not go to a beach.
11 would not go to the family cottage.
3 would go to neither a park nor a beach.
4 would go to neither a beach nor the family cottage.
6 would go to neither a park nor the family cottage.
2 would not go to a park or a beach or to the family cottage.
1 would go to all three places.

What is the total number of family members?

Recreational Mathematics

17. ***Number of Elements*** A universal set U consists of 12 elements. If sets A, B, and C are proper subsets of U and $n(U) = 12$, $n(A \cap B) = n(A \cap C) = n(B \cap C) = 6$, $n(A \cap B \cap C) = 4$, and $n(A \cup B \cup C) = 10$, determine

a) $n(A \cup B)$

b) $n(A' \cup C)$

c) $n(A \cap B)'$

SECTION 2.6 Infinite Sets

▲ *Georg Cantor, founder of set theory*

Which set is larger, the set of integers or the set of even integers? One might argue that because the set of even integers is a subset of the set of integers, the set of integers must be larger than the set of even integers. Yet both sets are infinite sets, so how can we determine which set is larger? This question puzzled mathematicians for centuries until 1874, when Georg Cantor developed a method of determining the cardinal number of an infinite set. In this section, we will discuss infinite sets and how to determine the number of elements in an infinite set.

Why *This Is Important* The concept of infinity and which sets contain more elements has led to the expansion and understanding of many mathematical and scientific concepts.

On page 45, we state that a finite set is a set in which the number of elements is zero or the number of elements can be expressed as a natural number. On page 46, we define a one-to-one correspondence. To determine the number of elements in a finite set, we can place the set in a one-to-one correspondence with a subset of the set of counting numbers. For example, the set $A = \{\#, ?, \$\}$ can be placed in one-to-one correspondence with set $B = \{1, 2, 3\}$, a subset of the set of counting numbers.

$$\begin{array}{rl} A = & \{\#,\ ?,\ \$\} \\ & \ \ \downarrow\ \ \downarrow\ \ \downarrow \\ B = & \{1,\ 2,\ 3\} \end{array}$$

Because the cardinal number of set B is 3, the cardinal number of set A is also 3. Any two sets, such as set A and set B, that can be placed in a one-to-one correspondence must have the same number of elements (therefore the same cardinality) and must be equivalent sets. Note that $n(A)$ and $n(B)$ both equal 3.

German mathematician Georg Cantor (1845–1918), known as the father of set theory, thought about sets that were not bounded. He called an unbounded set an *infinite set* and provided the following definition.

Definition: Infinite Set

An **infinite set** is a set that can be placed in a one-to-one correspondence with a proper subset of itself.

In Example 1, we use Cantor's definition of an infinite set to show that the set of counting numbers is infinite.

Example 1 *The Set of Natural Numbers*

Show that $N = \{1, 2, 3, 4, 5, \ldots, n, \ldots\}$ is an infinite set.

Solution To show that the set N is infinite, we establish a one-to-one correspondence between the counting numbers and a proper subset of itself. By removing the first element from the set of counting numbers, we get the set $\{2, 3, 4, 5, 6, \ldots\}$, which is a proper subset of the set of counting numbers. Now we establish the one-to-one correspondence.

$$\begin{array}{lcl} \text{Counting numbers} & = & \{1, 2, 3, 4, 5, \ldots, \quad n, \ldots\} \\ & & \;\;\downarrow\;\downarrow\;\downarrow\;\downarrow\;\downarrow \qquad\quad \downarrow \\ \text{Proper subset} & = & \{2, 3, 4, 5, 6, \ldots, n + 1, \ldots\} \end{array}$$

Note that for any number, n, in the set of counting numbers, its corresponding number in the proper subset is one greater, or $n + 1$. We have now shown the desired one-to-one correspondence, and thus the set of counting numbers is infinite. ■

Note in Example 1 that we showed the pairing of the general terms $n \rightarrow (n + 1)$. Showing a one-to-one correspondence of infinite sets requires showing the pairing of the general terms in the two infinite sets.

In the set of counting numbers, n represents the general term. For any other set of numbers, the general term will be different. The general term in any set should be written in terms of n such that when 1 is substituted for n in the general term, we get the first number in the set; when 2 is substituted for n in the general term, we get the second number in the set; when 6 is substituted for n in the general term, we get the sixth number in the set; and so on.

Consider the set $\{4, 9, 14, 19, \ldots\}$. Suppose we want to write the general term for this set (or sequence) of numbers. What would the general term be? The numbers differ by 5, so the general term will be of the form $5n$ plus or minus some number. Substituting 1 for n yields 5(1), or 5. Because the first number in the set is 4, we need to subtract 1 from the 5. Thus, the general term is $5n - 1$. Note that when $n = 1$, the value is $5(1) - 1$ or 4; when $n = 2$, the value is $5(2) - 1$ or 9; when $n = 3$, the value is $5(3) - 1$ or 14; and so on. Therefore, we write the set of numbers with the general term as

$$\{4, 9, 14, 19, \ldots, 5n - 1, \ldots\}$$

Now that you are aware of how to determine the general term of a set of numbers, we can do some more problems involving sets.

Example 2 *The Set of Even Numbers*

Show that the set of even counting numbers $\{2, 4, 6, 8, \ldots, 2n, \ldots\}$ is an infinite set.

Solution First create a proper subset of the set of even counting numbers by removing the first number from the set. Then establish a one-to-one correspondence.

$$\begin{array}{ll} \text{Even counting numbers:} & \{2, 4, 6, 8, \ldots, \quad 2n, \ldots\} \\ & \ \ \downarrow\ \downarrow\ \downarrow\ \downarrow \qquad\quad \downarrow \\ \text{Proper subset:} & \{4, 6, 8, 10, \ldots, 2n+2, \ldots\} \end{array}$$

A one-to-one correspondence exists between the two sets, so the set of even counting numbers is infinite. ■

Example 3 *The Set of Multiples of Five*

Show that the set $\{5, 10, 15, 20, \ldots, 5n, \ldots\}$ is an infinite set.

Solution

$$\begin{array}{ll} \text{Given set:} & \{5, 10, 15, 20, 25, \ldots, \quad 5n\ , \ldots\} \\ & \ \ \downarrow\ \ \downarrow\ \ \downarrow\ \ \downarrow\ \ \downarrow \qquad\quad \downarrow \\ \text{Proper subset:} & \{10, 15, 20, 25, 30, \ldots, 5n+5, \ldots\} \end{array}$$

Therefore, the given set is an infinite set. ■

Countable Sets

In his work with infinite sets, Cantor developed ideas on how to determine the cardinal number of an infinite set. He called the cardinal number of infinite sets "transfinite cardinal numbers" or "transfinite powers." He defined a set as ***countable*** if it is finite or if it can be placed in a one-to-one correspondence with the set of counting numbers. All infinite sets that can be placed in a one-to-one correspondence with the set of counting numbers have cardinal number ***aleph-null***, symbolized $\aleph_0$ (the first Hebrew letter, aleph, with a zero subscript, read "null").

Example 4 *The Cardinal Number of the Set of Even Numbers*

Show that the set of even counting numbers has cardinal number $\aleph_0$.

Solution In Example 2, we showed that the set of even counting numbers is infinite by setting up a one-to-one correspondence between the set and a proper subset of itself.

Now we will show that it is countable and has cardinality $\aleph_0$ by setting up a one-to-one correspondence between the set of counting numbers and the set of even counting numbers.

$$\begin{array}{ll} \text{Counting numbers:} & N = \{1, 2, 3, 4, \ldots, \ n, \ldots\} \\ & \qquad\ \ \downarrow\ \downarrow\ \downarrow\ \downarrow \qquad \downarrow \\ \text{Even counting numbers:} & E = \{2, 4, 6, 8, \ldots, 2n, \ldots\} \end{array}$$

For each number n in the set of counting numbers, its corresponding number is $2n$. Since we found a one-to-one correspondence between the set of counting numbers and the set of even counting numbers, the set of even counting numbers is countable. Thus, the cardinal number of the set of even counting numbers is $\aleph_0$; that is, $n(E) = \aleph_0$. As we mentioned earlier, the set of even counting numbers is an infinite set, since it can be placed in a one-to-one correspondence with a proper subset of itself. Therefore, the set of even counting numbers is both infinite and countable. ■

Definition: Cardinal Number of Infinite Sets

Any set that can be placed in a one-to-one correspondence with the set of counting numbers has **cardinal number** (or cardinality) $\aleph_0$ and is infinite and is countable.

Profile In Mathematics

Leopold Kronecker

Mathematician Leopold Kronecker (1823–1891), Cantor's former mentor, ridiculed Cantor's theories and prevented Cantor from gaining a position at the University of Berlin. Although Cantor's work on infinite sets is now considered a masterpiece, it generated heated controversy when originally published. Cantor's claim that the infinite set was unbounded offended the religious views of the time that God had created a complete universe that could not be wholly comprehended by humans. Eventually Cantor was given the recognition due him, but by then the criticism had taken its toll on his health. He had several nervous breakdowns and spent his last days in a mental hospital. See the Profile in Mathematics on page 43 for more information on Cantor.

Example 5 *The Cardinal Number of the Set of Odd Numbers*

Show that the set of odd counting numbers has cardinality $\aleph_0$.

Solution To show that the set of odd counting numbers has cardinality $\aleph_0$, we need to show a one-to-one correspondence between the set of counting numbers and the set of odd counting numbers.

$$\begin{array}{lll} \text{Counting numbers:} & N = & \{1, 2, 3, 4, 5, \ldots, \quad n \quad, \ldots\} \\ & & \ \ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow \qquad\quad \downarrow \\ \text{Odd counting numbers:} & O = & \{1, 3, 5, 7, 9, \ldots, 2n - 1, \ldots\} \end{array}$$

Since there is a one-to-one correspondence, the odd counting numbers have cardinality $\aleph_0$; that is, $n(O) = \aleph_0$. ■

We have shown that both the odd and the even counting numbers have cardinality $\aleph_0$. Merging the odd counting numbers with the even counting numbers gives the set of counting numbers, and we may reason that

$$\aleph_0 + \aleph_0 = \aleph_0$$

This result may seem strange, but it is true. What could such a statement mean? Well, consider a hotel with infinitely many rooms. If all the rooms are occupied, the hotel is, of course, full. If more guests appear, wanting accommodations, will they be turned away? The answer is *no*, for if the room clerk were to reassign each guest to a new room with a room number twice that of the present room, all the odd-numbered rooms would become unoccupied and there would be space for infinitely many more guests!

Cantor showed that there are different orders of infinity. Sets that are countable and have cardinal number $\aleph_0$ are the lowest order of infinity. Cantor showed that the set of integers and the set of rational numbers (fractions of the form p/q, where $q \neq 0$) are infinite sets with cardinality $\aleph_0$. He also showed that the set of real numbers (discussed in Chapter 5) could not be placed in a one-to-one correspondence with the set of counting numbers and that they have a higher order of infinity.

▲ *. . . where there's always room for one more. . .*

SECTION 2.6 Exercises

Warm Up Exercises

In exercises 1–2, fill in the blank with an appropriate word, phrase, or symbol(s).

1. A set that can be placed in a one-to-one correspondence with a proper subset of itself is called a(n) ________ set.

2. A set that is finite or can be placed in a one-to-one correspondence with the set of counting numbers is called a(n) ________ set.

Practice the Skills

In Exercises 3–12, show that the set is infinite by placing it in a one-to-one correspondence with a proper subset of itself. Be sure to show the pairing of the general terms in the sets.

3. $\{3, 4, 5, 6, 7, \ldots\}$

4. $\{30, 31, 32, 33, 34, \ldots\}$

5. $\{3, 5, 7, 9, 11, \ldots\}$

6. $\{20, 22, 24, 26, 28, \ldots\}$

7. $\{5, 9, 13, 17, 21, \ldots\}$

8. $\{6, 11, 16, 21, 26, \ldots\}$

9. $\{\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \frac{1}{10}, \ldots\}$

10. $\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots\}$

11. $\{\frac{4}{11}, \frac{5}{11}, \frac{6}{11}, \frac{7}{11}, \frac{8}{11}, \ldots\}$

12. $\{\frac{6}{13}, \frac{7}{13}, \frac{8}{13}, \frac{9}{13}, \frac{10}{13}, \ldots\}$

In Exercises 13–22, show that the set has cardinal number $\aleph_0$ by establishing a one-to-one correspondence between the set of counting numbers and the given set. Be sure to show the pairing of the general terms in the sets.

13. $\{3, 6, 9, 12, 15, \ldots\}$ **14.** $\{40, 41, 42, 43, 44, \ldots\}$

15. $\{4, 6, 8, 10, 12, \ldots\}$ **16.** $\{0, 2, 4, 6, 8, \ldots\}$

17. $\{2, 5, 8, 11, 14, \ldots\}$ **18.** $\{7, 11, 15, 19, 23, \ldots\}$

19. $\{\frac{1}{3}, \frac{1}{6}, \frac{1}{9}, \frac{1}{12}, \frac{1}{15}, \ldots\}$ **20.** $\{\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \ldots\}$

21. $\{\frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \ldots\}$ **22.** $\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \ldots\}$

Challenge Problems/Group Activities

In Exercises 23–26, show that the set has cardinality $\aleph_0$ by establishing a one-to-one correspondence between the set of counting numbers and the given set.

23. $\{1, 4, 9, 16, 25, \ldots\}$ **24.** $\{2, 4, 8, 16, 32, \ldots\}$

25. $\{3, 9, 27, 81, 243, \ldots\}$ **26.** $\{\frac{1}{3}, \frac{1}{6}, \frac{1}{12}, \frac{1}{24}, \frac{1}{48}, \ldots\}$

Recreational Mathematics

In Exercises 27–31, insert the symbol $<$, $>$, or $=$ in the shaded area to make a true statement.

27. $\aleph_0 \quad \aleph_0 + \aleph_0$ **28.** $2\aleph_0 \quad \aleph_0 + \aleph_0$

29. $2\aleph_0 \quad \aleph_0$ **30.** $\aleph_0 + 5 \quad \aleph_0 + 3$

31. $n(N) \quad \aleph_0$

32. There are a number of paradoxes (a statement that appears to be true and false at the same time) associated with infinite sets and the concept of infinity. One of these, called ***Zeno's Paradox,*** is named after the mathematician Zeno, born about 496 B.C. in Italy. According to Zeno's paradox, suppose Achelles starts out 1 meter behind a tortoise. Also, suppose Achelles walks 10 times as fast as the tortoise crawls. When Achelles reaches the point where the tortoise started, the tortoise is 1/10 of a meter ahead of Achelles; when Achelles reaches the point where the tortoise was 1/10 of a meter ahead, the tortoise is now 1/100 of a meter ahead; and so on. According to Zeno's Paradox, Achelles gets closer and closer to the tortoise but never catches up to the tortoise.

a) Do you believe the reasoning process is sound? If not, explain why not.

b) In actuality, if this situation were real, would Achelles ever pass the tortoise?

Internet/Research Activities

33. Do research to explain how Cantor proved that the set of rational numbers has cardinal number $\aleph_0$.

34. Do research to explain how it can be shown that the real numbers do not have cardinal number $\aleph_0$.

CHAPTER 2 *Summary*

Important Facts and Concepts		Examples and Discussion
Section 2.1		
Methods Used to Indicate a Set		
Description		Example 1, page 43
Roster Form		Examples 2–3, 5–7 pages 44, 45
Set-Builder Notation		Examples 4–6, pages 44–45
Symbol	*Meaning*	
$\in$	is an element of	Examples 4–6, pages 44–45
$\notin$	is not an element of	
$n(A)$	number of elements in set A	
$\emptyset$ or $\{\ \}$	the empty set	
U	the universal set	

Section 2.2

Symbol	Meaning	
$\subseteq$	is a subset of	Examples 1 and 3, pages 52–53
$\not\subseteq$	is not a subset of	
$\subset$	is a proper subset of	
$\not\subset$	is not a proper subset of	

Number of distinct subsets of a finite set with n elements is 2^n. — Examples 4 and 5, pages 54

Section 2.3

Symbol	Meaning	
$'$	complement	Examples 1, 3–6, pages 57, 58–61
$\cap$	intersection	Examples 2, 3, 6, pages 58–61
$\cup$	union	Examples 4–6, pages 57–61
$-$	difference of two sets	Example 9, page 62
$\times$	cartesian product	Example 10, page 63

And is generally interpreted to mean *intersection*. — Examples 7 and 8, pages 61–62

Or is generally interpreted to mean *union*. — Examples 7 and 8, pages 61–62

For any sets A and B, — Examples 7 and 8, pages 61–62

$n(A \cup B) = n(A) + n(B) - n(A \cap B)$

Section 2.4

De Morgan's Laws — Example 3, page 71

$(A \cap B)' = A' \cup B'$

$(A \cup B)' = A' \cap B'$

Section 2.6

An **infinite set** is a set that can be placed in a one-to-one correspondence with a proper subset of itself. — Examples 1–5, pages 84–86, Discussion pages 83–84

$\aleph_0$ aleph-null — Examples 4–5, pages 85–86

Countable sets — Examples 4–5, pages 85–86

CHAPTER 2 Review Exercises

2.1, 2.2, 2.3, 2.4, 2.6

In Exercises 1–14, state whether each statement is true or false. If false, give a reason.

1. The set of cities located in the state of Indiana is a well-defined set.
2. The set of the three best movies is a well-defined set.
3. maple $\in$ {oak, elm, maple, sycamore}
4. $\{\ \} \subset \varnothing$
5. $\{3, 6, 9, 12, \dots\}$ and $\{2, 4, 6, 8, \dots\}$ are disjoint sets.
6. $\{a, b, c, 1, 2\}$ is an example of a set in roster form.
7. {purple, green, yellow} = {green, pink, yellow}
8. {apple, orange, banana, pear} is equivalent to {tomato, corn, spinach, radish}.
9. If $A = \{a, e, i, o, u\}$, then $n(A) = 5$.
10. $A = \{1, 3, 5, 7, \dots\}$ is a countable set.
11. $A = \{1, 4, 7, 10, \dots, 31\}$ is a finite set.
12. $\{2, 5, 7\} \subseteq \{2, 5, 7, 10\}$.
13. $\{x \mid x \in N \text{ and } 3 < x \le 9\}$ is a set in set-builder notation.
14. $\{x \mid x \in N \text{ and } 2 < x \le 12\} \subseteq \{1, 2, 3, 4, 5, \dots, 20\}$

In Exercises 15–18, express each set in roster form.

15. Set A is the set of odd natural numbers between 5 and 16.

16. Set B is the set of states that border Kansas.

17. $C = \{x \mid x \in N \text{ and } x < 162\}$

18. $D = \{x \mid x \in N \text{ and } 8 < x \leq 80\}$

In Exercises 19–22, express each set in set-builder notation.

19. Set A is the set of natural numbers between 50 and 150.

20. Set B is the set of natural numbers greater than 42.

21. Set C is the set of natural numbers less than 7.

22. Set D is the set of natural numbers between 27 and 51, inclusive.

In Exercises 23–26, express each set with a written description.

23. $A = \{x \mid x$ is a capital letter of the English alphabet from E through M inclusive$\}$

24. $B = \{$penny, nickel, dime, quarter, half-dollar$\}$

25. $C = \{a, b, c\}$

26. $D = \{x \mid 3 \leq x < 9\}$

In Exercises 27–36, let

$$U = \{1, 2, 3, 4, \ldots, 10\}$$
$$A = \{1, 3, 5, 7\}$$
$$B = \{3, 7, 9, 10\}$$
$$C = \{1, 7, 10\}$$

Determine the following.

27. $A \cap B$

28. $A \cup B'$

29. $A' \cap B$

30. $(A \cup B)' \cup C$

31. $A - B$

32. $A - C'$

33. $A \times C$

34. $B \times A$

35. The number of subsets of set B

36. The number of proper subsets of set A

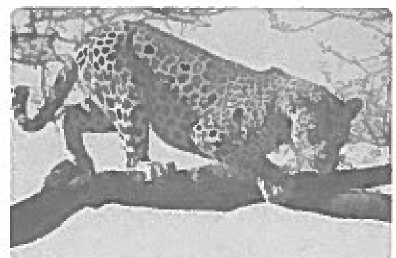
37. For the following sets, construct a Venn diagram and place the elements in the proper region.

$U = \{$lion, tiger, leopard, cheetah, puma, lynx, panther, jaguar$\}$

$A = \{$tiger, puma, lynx$\}$

$B = \{$lion, tiger, jaguar, panther$\}$

$C = \{$tiger, lynx, cheetah, panther$\}$



In Exercises 38–43, use Fig. 2.29 to determine the sets.

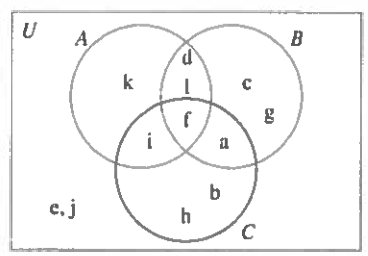


Figure 2.29

38. $A \cup B$

39. $A \cap B'$

40. $A \cup B \cup C$

41. $A \cap B \cap C$

42. $(A \cup B) \cap C$

43. $(A \cap B) \cup C$

Construct a Venn diagram to determine whether the following statements are true for all sets A, B, and C.

44. $(A' \cup B')' = A \cap B$

45. $(A \cup B') \cup (A \cup C') = A \cup (B \cap C)'$

In Exercises 46–51, use the following table, which shows the amount of sugar, in grams (g) and caffeine, in milligrams (mg), in an 8-oz serving of selected beverages. Let the beverages listed represent the universal set.

Beverage	Sugar (grams, g)	Caffeine (milligrams, mg)
Mountain Dew	31	37
Coca-Cola	27	23
Pepsi	27	25
Sprite	26	0
Brewed coffee	0	108
Brewed tea	0	47
Orange juice	24	0
Grape juice	40	0
Gatorade	14	0
Red Bull	26	76
Vitamin Water	13	17
Water	0	0

Source: International Food Information Council

Let A be the set of beverages that contain at least 20 g of sugar.

Let B be the set of beverages that contain at least 20 mg of caffeine.

Indicate in Fig 2.30 in which region, I–IV, each of the following beverages belongs.

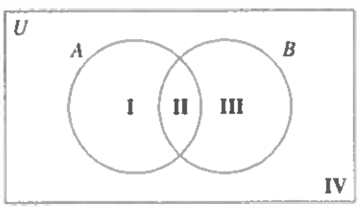


Figure 2.30

46. Pepsi

47. Brewed coffee

48. Orange juice

49. Vitamin Water

50. Gatorade

51. Mountain Dew

52. Red Bull

2.5

53. *Pizza Survey* A pizza chain was willing to pay $1 to each person interviewed about his or her likes and dislikes of types of pizza crust. Of the people interviewed, 200 liked thin crust, 270 liked thick crust, 70 liked both, and 50 did not like pizza at all. What was the total cost of the survey?

54. *Shopping Preferences* Visitors to a shopping mall in Atlanta, Georgia, were surveyed to determine their preference for shopping in wholesale warehouse stores. The following information was determined.

58 shopped in BJ's Wholesale Club.
49 shopped in Sam's Club.
45 shopped in Costco.
15 shopped in BJ's Wholesale Club and Sam's Club.
16 shopped in BJ's Wholesale Club and Costco.
12 shopped in Sam's Club and Costco.
5 shopped in all three stores.
17 did not shop in any of the three stores.

Construct a Venn diagram and then determine how many people

a) completed the survey.

b) shopped only in BJ's Wholesale Club.

c) shopped in BJ's Wholesale Club and Sam's Club, but not Costco.

d) shopped in BJ's Wholesale Club or Costco, but not Sam's Club.

55. *TV Choices TV Guide* surveyed 510 subscribers asking which of the following three crime investigation shows they watched on a regular basis: *CSI:Crime Scene Investigation, CSI:Miami,* and *CSI:NY.* The results of the 510 questionnaires that were returned showed that

175 watched *CSI:NY.*
227 watched *CSI:Miami.*
285 watched *CSI:Crime Scene Investigation.*
100 watched *CSI:NY* and *CSI:Miami.*
96 watched *CSI:NY* and *CSI:Crime Scene Investigation.*
87 watched *CSI:Miami* and *CSI:Crime Scene Investigation.*
59 watched all three shows.

Construct a Venn diagram and determine how many people

a) watched only *CSI:NY.*

b) watched exactly one of these shows.

c) watched *CSI:Miami* and *CSI:Crime Scene Investigation,* but not *CSI:NY.*

d) watched *CSI:NY* or *CSI:Crime Scene Investigation,* but not *CSI:Miami.*

e) watched exactly two of these shows.

▲ *Actors from CSI:NY*

2.6

In Exercises 56 and 57, show that the sets are infinite by placing each set in a one-to-one correspondence with a proper subset of itself.

56. $\{2, 4, 6, 8, 10, \dots\}$

57. $\{3, 5, 7, 9, 11, \dots\}$

In Exercises 58 and 59, show that each set has cardinal number $\aleph_0$ by setting up a one-to-one correspondence between the set of counting numbers and the given set.

58. $\{5, 8, 11, 14, 17, \dots\}$

59. $\{4, 9, 14, 19, 24, \dots\}$

CHAPTER 2 *Test*

In Exercises 1–8, state whether each is true or false. If the statement is false, explain why.

1. $\{2, y, \triangle, \$\}$ is equivalent to $\{p, \#, 4, \square\}$.

2. $\{3, 5, 9, h\} = \{9, 5, 3, j\}$

3. $\{\text{star, moon, sun}\} \subset \{\text{star, moon, sun, planet}\}$

4. $\{7\} \subseteq \{x \mid x \in N \text{ and } x < 7\}$

5. $\{p, q, r, s\}$ has 15 subsets.

6. If $A \cap B = \{\ \}$, then A and B are disjoint sets.

7. For any set A, $A \cup A' = \{\ \}$.

8. For any set A, $A \cap U = A$.

In Exercises 9 and 10, use set

$$A = \{x \mid x \in N \text{ and } x < 10\}$$

9. Write set A in roster form.

10. Write a description of set A.

In Exercises 11–16, use the following information.

$$U = \{3, 5, 7, 9, 11, 13, 15\}$$
$$A = \{3, 5, 7, 9\}$$
$$B = \{7, 9, 11, 13\}$$
$$C = \{3, 11, 15\}$$

Determine the following.

11. $A \cap B$

12. $A \cup C'$

13. $A \cap (B \cap C')$

14. $n(A \cap B')$

15. $A - B$

16. $A \times C$

17. Using the sets provided for Exercises 11–16, draw a Venn diagram illustrating the relationship among the sets.

18. Use a Venn diagram to determine whether

$$A \cap (B \cup C') = (A \cap B) \cup (A \cap C')$$

for all sets A, B, and C. Show your work.

19. ***Water Activities*** A survey of 155 residents of Lake Placid were asked what kind of water activities they participated in on a daily basis during the summer months. The following information was determined.

107 swam.
90 sailed.
76 water skied.
57 swam and sailed.
54 swam and water skied.
52 sailed and water skied.
35 swam, sailed, and water skied.

Construct a Venn diagram and then determine the number of residents who participated in

a) exactly one of these activities.
b) none of these activities.
c) at least two of these activities.
d) swimming and sailing, but not water skiing.
e) swimming or sailing, but not water skiing.
f) only water skiing.

20. Show that the following set is infinite by setting up a one-to-one correspondence between the set and a proper subset of itself.

$$\{7, 8, 9, 10, \dots\}$$

GROUP PROJECTS

Selecting a Family Pet

1. The Wilcox family is considering buying a dog. They have established several criteria for the family dog: It must be one of the breeds listed in the table, must not shed, must be less than 16 in. tall, and must be good with children.

   a) Using the information in the table,* construct a Venn diagram in which the universal set is the dogs listed. Indicate the set of dogs to be placed in each region of the Venn diagram.

   b) From the Venn diagram constructed in part (a), determine which dogs will meet the criteria set by the Wilcox family. Explain.

Breed	Sheds	Less than 16 in.	Good with children
Airedale	no	no	no
Basset hound	yes	yes	yes
Beagle	yes	yes	yes
Border terrier	no	yes	yes
Cairn terrier	no	yes	no
Cocker spaniel	yes	yes	yes
Collie	yes	no	yes
Dachshund	yes	yes	no
Poodle, miniature	no	yes	no
Schnauzer, miniature	no	yes	no
Scottish terrier	no	yes	no
Wirehaired fox terrier	no	yes	no

Classification of the Domestic Cat

2. Read the Mathematics Today feature on page 53. Do research and indicate the name of the following groupings to which the domestic cat belongs.

   a) Kingdom

   b) Phylum

   c) Class

   d) Order

   e) Family

   f) Genus

   g) Species

Who Lives Where

3. On Diplomat Row, an area of Washington, DC, there are five houses. Each owner is a different nationality, each has a different pet, each has a different favorite food, each has a different favorite drink, and each house is painted a different color.

   The green house is directly to the right of the ivory house.
   The Senegalese has the red house.
   The dog belongs to the Spaniard.
   The Afghanistani drinks tea.
   The person who eats cheese lives next door to the fox.
   The Japanese eats fish.
   Milk is drunk in the middle house.
   Apples are eaten in the house next to the horse.
   Ale is drunk in the green house.
   The Norwegian lives in the first house.
   The peach eater drinks whiskey.
   Apples are eaten in the yellow house.
   The banana eater owns a snail.
   The Norwegian lives next door to the blue house.

   For each house find

   a) the color.

   b) the nationality of the occupant.

   c) the owner's favorite food.

   d) the owner's favorite drink.

   e) the owner's pet.

   f) Finally, the crucial question is: Does the zebra's owner drink vodka or ale?

*The information is a collection of the opinions of an animal psychologist, Dr. Daniel Tortora, and a group of veterinarians.