Quiz 10

1. Evaluate
$$\lim_{x\to 0} \frac{e^{2x} - 2e^x + 1}{\cos(2x) - 1}.$$
We apply L'Hôpital's Rule:
$$\begin{aligned}
\lim_{x\to 0} \frac{e^{2x} - 2e^x + 1}{\cos(2x) - 1} &= \lim_{x\to 0} \frac{\frac{\mathrm{d}}{\mathrm{dx}}\left(e^{2x} - 2e^x + 1\right)}{\frac{\mathrm{d}}{\mathrm{dx}}\left(\cos(2x) - 1\right)} \\
&= \lim_{x\to 0} \frac{2e^{2x} - 2e^x}{-2\sin(2x)} \\
&= \lim_{x\to 0} \frac{\frac{\mathrm{d}}{\mathrm{dx}}\left(2e^{2x} - 2e^x\right)}{\frac{\mathrm{d}}{\mathrm{dx}}\left(-2\sin(2x)\right)} \\
&= \lim_{x\to 0} \frac{4e^{2x} - 2e^x}{-4\cos(2x)} \\
&= \frac{4e^0 - 2e^0}{-4\cos(0)} \\
&= \frac{2}{-4} = -\frac{1}{2}
\end{aligned}$$

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2. Sketch the graph of $f(x)$, given
$$f(x) = \frac{4x-3}{2x+4} \qquad f'(x) = \frac{11}{2(x+2)^2} \qquad f''(x) = -\frac{11}{(x+2)^3}$$

We are after the intervals of concavity, intervals of increase/decrease, transition points, and asymptotic behavior of this function.

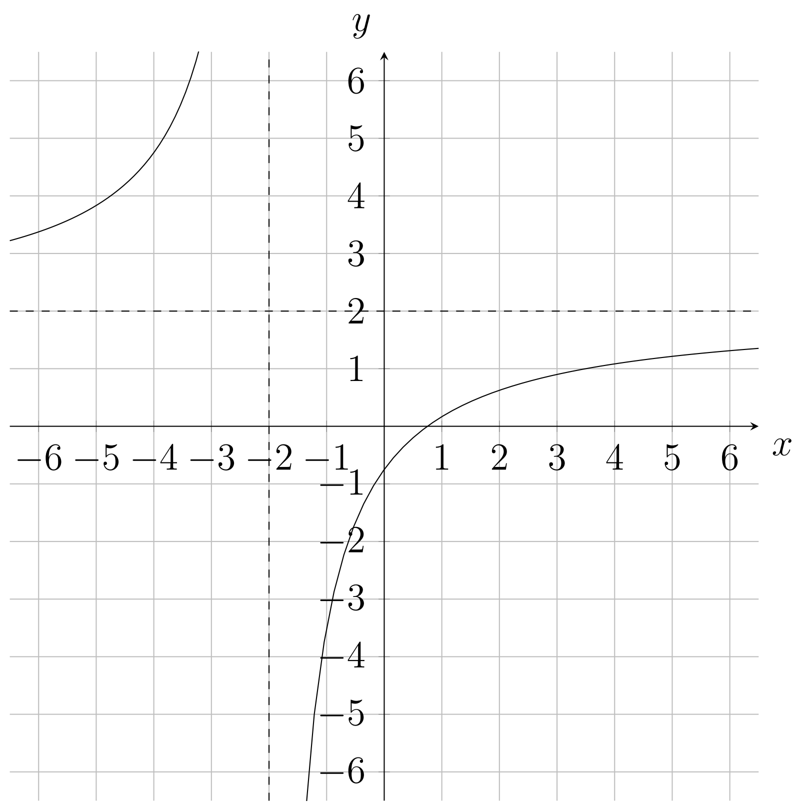
The critical values of f are where the derivative is either zero or undefined. Certainly the derivative is never zero, so it suffices to consider where it is undefined. The derivative $f'(x) = \frac{11}{2}(x+2)^{-2}$ is undefined at $x = -2$, so this is our only critical point. Furthermore, anywhere the derivative is defined, it is positive, so our function is increasing on $(-\infty, -2) \cup (-2, \infty)$.

Now, our potential inflection points are where the second derivative is zero or undefined. Similarly to before, our second derivative is never zero, so it suffices to consider where it undefined. Our second derivative $f''(x) = -11(x+2)^{-3}$ is undefined at $x = -2$, so this is our only potential inflection point. Now, we check points on either side: $f''(-3) = 11 > 0$, and $f''(-1) = -11 < 0$. So f is concave *up* to the left of -2 and concave *down* to the right of -2.

Our function f has a vertical asymptote at $x = -2$. To get our horizontal asymptotes, we consider

$$\lim_{x\to\infty} \frac{4x-3}{2x+4} = \frac{4}{2} = 2 \qquad\qquad \lim_{x\to-\infty} \frac{2x+4}{3x-3} = \frac{4}{2} = 2,$$

Where the limits may be evaluated using the leading coefficient test or using L'Hôpital's Rule. Using this information, we may graph our function as below.



Quiz 10

1. Evaluate
$$\lim_{x\to 0}\frac{\cos(3x)-1}{e^{3x}-3e^{x}+1}.$$
This limit does *not* yield an indeterminate form:
$$\lim_{x\to 0}\frac{\cos(3x)-1}{e^{3x}-3e^{x}+1}=\frac{\cos(0)-1}{e^{0}-3e^{0}+1}=\frac{0}{-1}=0.$$

2. Sketch the graph of $f(x)$, given

$$f(x) = \frac{2x+4}{3x-3} \qquad f'(x) = -\frac{2}{(x-1)^2} \qquad f''(x) = \frac{4}{(x-1)^3}$$

We are after the intervals of concavity, intervals of increase/decrease, transition points, and asymptotic behavior of this function.

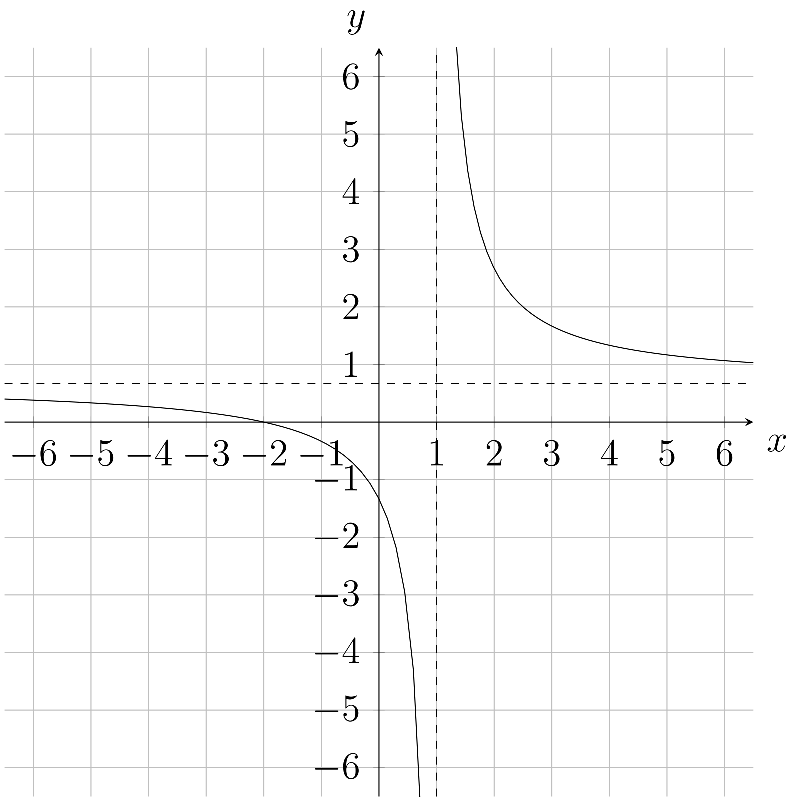
The critical values of f are where the derivative is either zero or undefined. Certainly the derivative is never zero, so it suffices to consider where it is undefined. The derivative $f'(x) = -2(x-1)^{-2}$ is undefined at $x = 1$, so this is our only critical point. Furthermore, anywhere the derivative is defined, it is negative, so our function is decreasing on $(-\infty, 1) \cup (1, \infty)$.

Now, our potential inflection points are where the second derivative is zero or undefined. Similarly to before, our second derivative is never zero, so it suffices to consider where it undefined. Our second derivative $f''(x) = 4(x-1)^{-3}$ is undefined at $x = 1$, so this is our only potential inflection point. Now, we check points on either side: $f''(0) = -4 < 0$, and $f''(2) = 4 > 0$. So f is concave *up* to the right of 1 and concave *down* to the left of 1.

Our function f has a vertical asymptote at $x = 1$. To get our horizontal asymptotes, we consider

$$\lim_{x\to\infty} \frac{2x+4}{3x-3} = \frac{2}{3} \qquad\qquad \lim_{x\to-\infty} \frac{2x+4}{3x-3} = \frac{2}{3},$$

Where the limits may be evaluated using the leading coefficient test or using L'Hôpital's Rule. Using this information, we may graph our function as below.


y
x
6
5
4
3
2
1
−1
−2
−3
−4
−5
−6
−6 −5 −4 −3 −2 −1
1 2 3 4 5 6