1. Evaluate
$$\lim_{x\to-\infty}\left(x+\sqrt{x^2+4x+5}\right).$$

Solution. Since plugging in $-\infty$ yields an indeterminate form of $-\infty+\infty$, we have to try harder. In this case, multiplying by the conjugate (over itself) helps us:

$$\begin{aligned}\lim_{x\to-\infty}\left(x+\sqrt{x^2+4x+5}\right) &= \lim_{x\to-\infty}\left(x+\sqrt{x^2+4x+5}\right)\cdot\frac{x-\sqrt{x^2+4x+5}}{x-\sqrt{x^2+4x+5}}\\ &= \lim_{x\to-\infty}\frac{x^2-(x^2+4x+5)}{x-\sqrt{x^2+4x+5}}\\ &= \lim_{x\to-\infty}\frac{-4x-5}{x-\sqrt{x^2+4x+5}}.\end{aligned}$$

Unfortunately, plugging in $-\infty$ still yields an indeterminate form, but we note that the 'degree' of the numerator and the denominator are the same, where we consider $\sqrt{x^2+4x+5}$ to have 'degree' 1. We then divide the numerator and denominator by x^1, using that for negative values of x, $x^1=-\sqrt{x^2}$.

$$\begin{aligned}\lim_{x\to-\infty}\frac{-4x-5}{x-\sqrt{x^2+4x+5}} &= \lim_{x\to-\infty}\frac{\frac{-4x}{x}-\frac{5}{x}}{\frac{x}{x}-\frac{\sqrt{x^2+4x+5}}{-\sqrt{x^2}}}\\ &= \lim_{x\to-\infty}\frac{-4-\frac{5}{x}}{1+\sqrt{1+\frac{4}{x}+\frac{5}{x^2}}}\end{aligned}$$

Now, plugging in $-\infty$ gives us the answer we so desire:

$$\lim_{x\to-\infty}\frac{-4-\frac{5}{x}}{1+\sqrt{1+\frac{4}{x}+\frac{5}{x^2}}}=\frac{-4}{1+\sqrt{1}}=-2.$$

$$\boxed{-2}$$

2. Find all horizontal asymptotes of
$$f(x)=\frac{2e^x-4}{4-e^{2x}}.$$

Solution. Horizontal asymptotes are just limits at $\pm\infty$, so we find the limit of $f(x)$ as x goes to either ∞ or $-\infty$. First, we see that by plugging in,

$$\lim_{x\to-\infty}\frac{2e^x-4}{4-e^{2x}}=\frac{-4}{4}=-1,$$

so one of our horizontal asymptotes is $y = -1$. To find our other limit, first note that plugging in ∞ yields an indeterminate form. Then, we apply a method similar to that of problem 1 and divide both the numerator and the denominator by e^x.

$$\lim_{x\to\infty} \frac{2e^x - 4}{4 - e^{2x}} = \lim_{x\to\infty} \frac{2 - \frac{4}{e^x}}{\frac{4}{e^x} - e^x}.$$

Now, plugging in does not yield an indeterminate form, and we have

$$\begin{aligned} \lim_{x\to\infty} \frac{2e^x - 4}{4 - e^{2x}} &= \lim_{x\to\infty} \frac{2 - \frac{4}{e^x}}{\frac{4}{e^x} - e^x} \\ &= \frac{2}{-\infty} \\ &= 0. \end{aligned}$$

Thus we have *two* horizontal asymptotes,

$$\boxed{y = -1, 0}$$

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3. **Using the limit definition**, find the equation of the tangent line of $f(x) = 1 - x^2$ at $x = 3$.

Solution. The tangent line to $f(x)$ at $x = 3$ is the line that has slope $f'(3)$ and goes through the point $(3, f(3))$. We first find $f'(3)$ using the limit definition.

$$\begin{aligned} f'(3) &= \lim_{h\to 0} \frac{f(3+h) - f(3)}{h} \\ &= \lim_{h\to 0} \frac{1 - (3+h)^2 - (1 - 3^2)}{h} \\ &= \lim_{h\to 0} \frac{-9 - 6h - h^2 + 9}{h} \\ &= \lim_{h\to 0} \frac{-h(6+h)}{h} \\ &= \lim_{h\to 0} -(6+h) \\ &= -6. \end{aligned}$$

Now, we just use this slope along with our point $(3, f(3)) = (3, -8)$ to get our line:

$$\boxed{y + 8 = -6(x - 3)}$$