1. Evaluate
$$\lim_{x\to\infty}\left(x-\sqrt{x^2+4x+5}\right).$$

Solution. Since plugging in $-\infty$ yields an indeterminate form of $-\infty+\infty$, we have to try harder. In this case, multiplying by the conjugate (over itself) helps us:

$$\begin{aligned}\lim_{x\to\infty}\left(x-\sqrt{x^2+4x+5}\right) &= \lim_{x\to\infty}\left(x-\sqrt{x^2+4x+5}\right)\cdot\frac{x+\sqrt{x^2+4x+5}}{x+\sqrt{x^2+4x+5}}\\ &= \lim_{x\to\infty}\frac{x^2-(x^2+4x+5)}{x+\sqrt{x^2+4x+5}}\\ &= \lim_{x\to\infty}\frac{-4x-5}{x+\sqrt{x^2+4x+5}}.\end{aligned}$$

Unfortunately, plugging in ∞ still yields an indeterminate form, but we note that the 'degree' of the numerator and the denominator are the same, where we consider $\sqrt{x^2+4x+5}$ to have 'degree' 1. We then divide the numerator and denominator by x^1, using that for positive values of x, $x^1=\sqrt{x^2}$.

$$\begin{aligned}\lim_{x\to\infty}\frac{-4x-5}{x+\sqrt{x^2+4x+5}} &= \lim_{x\to\infty}\frac{\frac{-4x}{x}+\frac{5}{x}}{\frac{x}{x}+\frac{\sqrt{x^2+4x+5}}{\sqrt{x^2}}}\\ &= \lim_{x\to\infty}\frac{-4-\frac{5}{x}}{1+\sqrt{1+\frac{4}{x}+\frac{5}{x^2}}}\end{aligned}$$

Now, plugging in ∞ gives us the answer we so desire:

$$\lim_{x\to\infty}\frac{-4-\frac{5}{x}}{1+\sqrt{1+\frac{4}{x}+\frac{5}{x^2}}}=\frac{-4}{1+\sqrt{1}}=-2.$$

$$\boxed{-2}$$

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2. Find all horizontal asymptotes of
$$f(x)=\frac{6+2e^x}{e^{2x}-9}.$$

Solution. Horizontal asymptotes are just limits at $\pm\infty$, so we find the limit of $f(x)$ as x goes to either ∞ or $-\infty$. First, we see that by plugging in,

$$\lim_{x\to-\infty}\frac{6+2e^x}{e^{2x}-9}=\frac{6}{-9}=-\frac{2}{3},$$

so one of our horizontal asymptotes is $y = -2/3$. To find our other limit, first note that plugging in ∞ yields an indeterminate form. Then, we apply a method similar to that of problem 1 and divide both the numerator and the denominator by e^x.

$$\lim_{x\to\infty} \frac{6+2e^x}{e^{2x}-9} = \lim_{x\to\infty} \frac{\frac{6}{e^x}+2}{e^x - \frac{9}{e^x}}.$$

Now, plugging in does not yield an indeterminate form, and we have

$$\begin{aligned} \lim_{x\to\infty} \frac{6+2e^x}{e^{2x}-9} &= \lim_{x\to\infty} \frac{\frac{6}{e^x}+2}{e^x - \frac{9}{e^x}} \\ &= \frac{2}{+\infty} \\ &= 0. \end{aligned}$$

Thus we have *two* horizontal asymptotes,

$$\boxed{y = -2/3, 0}$$

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3. **Using the limit definition**, find the equation of the tangent line of $f(x) = x^2 + 9$ at $x = -3$.

Solution. The tangent line to $f(x)$ at $x = -3$ is the line that has slope $f'(-3)$ and goes through the point $(-3, f(-3))$. We first find $f'(-3)$ using the limit definition.

$$\begin{aligned} f'(-3) &= \lim_{h\to 0} \frac{f(-3+h)-f(-3)}{h} \\ &= \lim_{h\to 0} \frac{(-3+h)^2+9-((-3)^2+9)}{h} \\ &= \lim_{h\to 0} \frac{-6h+h^2}{h} \\ &= \lim_{h\to 0} \frac{h(h-6)}{h} \\ &= \lim_{h\to 0} h-6 \\ &= -6. \end{aligned}$$

Now, we just use this slope along with our point $(-3, f(-3)) = (-3, 18)$ to get our line:

$$\boxed{y - 18 = -6(x+3)}$$