1. (2 points) Find the point on the graph $y = \frac{1}{2}x^2$ closest to the point $(0, 1)$.

   Solution. The distance between a point $(x, \frac{1}{2}x^2)$ on the curve $y = \frac{1}{2}x^2$ and the point $(0, 1)$ is given by

$$\begin{aligned} d(x) &= \sqrt{(x-0)^2 + (\frac{1}{2}x^2 - 1)^2} \\ &= \sqrt{x^2 + \frac{1}{4}x^4 - x^2 + 1} \\ &= \sqrt{\frac{1}{4}(x^4 + 4)} \\ &= \frac{1}{2}\sqrt{x^4 + 4}. \end{aligned}$$

   Now, we find the absolute minimum of d on its domain $(-\infty, \infty)$.

$$\begin{aligned} d'(x) &= \frac{d}{dx}\left(\frac{1}{2}\sqrt{x^4 + 4}\right) \\ &= \frac{1}{4}\frac{4x^3}{\sqrt{x^4 + 4}} \\ &= \frac{x^3}{\sqrt{x^4 + 4}}. \end{aligned}$$

   Since the denominator $\sqrt{x^4 + 4}$ is always positive, our only critical point is where the numerator is 0, meaning $x = 0$, and this point must represent an absolute minimum of d. Thus the closest point to $(0, 1)$ on $y = \frac{1}{2}x^2$ is $(0, 0)$.

$$\boxed{(0, 0)}$$

2. (4 points) Sketch the graph of $f(x)$, given

$$f(x) = \frac{-2x+4}{2x+2} \qquad f'(x) = \frac{-14}{(2x+2)^2} \qquad f''(x) = \frac{56}{(2x+2)^3}$$

Be sure to include asymptotes, x- and y-intercepts, where f is increasing/decreasing and concave up/down.

Solution. First, the horizontal asymptote(s):

$$\lim_{x\to-\infty} \frac{-2x+4}{2x+2} = \frac{-2}{2} = -1 \qquad \text{and} \qquad \lim_{x\to+\infty} \frac{-2x+4}{2x+2} = \frac{-2}{2} = -1,$$

using either L'Hospital's Rule or the ratio of the leading coefficients. Now, the only candidate for a vertical asymptote is when $2x+2=0$, meaning $x=-1$. Indeed,

$$\lim_{x\to-1^-} \frac{-2x+4}{2x+2} = \frac{6}{0^-} = -\infty \qquad \text{and} \qquad \lim_{x\to-1^+} \frac{-2x+4}{2x+2} = \frac{6}{0^+} = +\infty,$$

so $x=-1$ does represent a vertical asymptote. For our x-intercept, we have $f(x)=0$ when $-2x+4=0$, so $(2,0)$ is our x-intercept. For our y-intercept, we have $f(0)=2$, so $(0,2)$ is our y-intercept.

Both f' and f'' are undefined at $x=-1$, so that point should go on both the f' and f'' sign lines. Checking the sign of both f' and f'' on either side of this point lets us finish our sign lines. Together with our intercepts and asymptotes, we can sketch our graph.