Quiz 9

1. Find each value of x where $f(x) = \ln(x) + 100x + \frac{1}{2}x^2$ has an inflection point.

   We first find the second derivative of f:

$$\begin{aligned} f'(x) &= \frac{\mathrm{d}}{\mathrm{dx}} f(x) \\ &= \frac{1}{x} + 100 + x \\ f''(x) &= \frac{\mathrm{d}}{\mathrm{dx}} f'(x) \\ &= -\frac{1}{x^2} + 1. \end{aligned}$$

   We consider those values of x where $f''(x)$ is zero or undefined. Here, $f''(x) = 0$ when $x = 1$, and $f''(x)$ is undefined when $x = 0$. Of course, the function is undefined at $x = 0$, so we don't consider it.

   Now, we note that to the left of $x = 1$ the second derivative of f is negative, while to the right of $x = 1$ the second derivative is positive. So $x = 1$ indeed represents a point of inflection.

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2. Let $g(x) = \dfrac{x^4 - 16}{x + 2}$. Note that

$$g'(x) = 3x^2 - 4x + 4 \qquad\qquad g''(x) = 6x - 4.$$

   Find each interval where g is increasing and concave down.

   Our goal is to find the intervals in which g' is positive and the intervals where g'' is negative and compute their intersection.

   g' is an upward-opening parabola. Furthermore, as the *discriminant* of g', $b^2 - 4ac = -32$, is negative, we know that g' has no *real roots*, ie. it never touches the x-axis. So g' is always positive.

   Now, to see where g'' is negative, we have

$$6x - 4 < 0 \iff x < \frac{2}{3},$$

so g is concave down on $(-\infty, \frac{2}{3})$. Together, g is increasing and concave down on $(-\infty, \frac{2}{3})$.

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3. Evaluate

$$\lim_{x\to\infty} \frac{e^x}{x^2}.$$

Note that, naively,

$$\lim_{x\to\infty} \frac{e^x}{x^2} = \frac{\infty}{\infty},$$

so we may apply L'Hospital's Rule. Thus, we have

$$\lim_{x\to\infty} \frac{e^x}{x^2} = \lim_{x\to\infty} \frac{\frac{\mathrm{d}}{\mathrm{dx}}\, e^x}{\frac{\mathrm{d}}{\mathrm{dx}}\, x^2} = \lim_{x\to\infty} \frac{e^x}{2x}.$$

Again, our limit works out to $\dfrac{\infty}{\infty}$, so we may apply L'Hospital's Rule again:

$$\lim_{x\to\infty} \frac{e^x}{2x} = \lim_{x\to\infty} \frac{\frac{\mathrm{d}}{\mathrm{dx}}\, e^x}{\frac{\mathrm{d}}{\mathrm{dx}}\, 2x} = \lim_{x\to\infty} \frac{e^x}{2} = \infty.$$

Quiz 9

1. Find each value of x where $f(x) = e^x - 100x - \frac{1}{2}x^2$ has an inflection point.

   We first find the second derivative of f:

$$\begin{aligned} f'(x) &= \frac{\mathrm{d}}{\mathrm{dx}} f(x) \\ &= e^x - 100 - x \\ f''(x) &= \frac{\mathrm{d}}{\mathrm{dx}} f'(x) \\ &= e^x - 1 \end{aligned}$$

   We consider those values of x where $f''(x)$ is zero or undefined. Here, $f''(x)$ is defined for all x, and $f''(x) = 0$ when $x = 0$. Now, we note that to the left of $x = 0$ the second derivative of f is negative, while to the right of $x = 0$ the second derivative is positive. So $x = 0$ indeed represents a point of inflection.

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2. Let $g(x) = \dfrac{x^4 - 16}{x - 2}$. Note that

$$g'(x) = 3x^2 + 4x + 4 \qquad\qquad g''(x) = 6x + 4.$$

   Find each interval where g is increasing and concave up.

   Our goal is to find the intervals in which g' is positive and the intervals where g'' is negative and then compute their intersection.

   g' is an upward opening parabola. Furthermore, as the *discriminant* of g', $b^2 - 4ac = -32$, is negative, we know that g' has no *real roots*, ie. it never touches the x-axis. As such, g' is always positive.

   Now, to see where g'' is positive, we have

$$6x + 4 < 0 \iff x < -\frac{2}{3},$$

   so g is concave up on $(-\infty, -\frac{2}{3})$. Together, g is both increasing and concave up on $(-\infty, -\frac{2}{3})$.

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3. Evaluate

$$\lim_{x\to\infty} \frac{\ln(x)}{x^2}.$$

Note that, naively,

$$\lim_{x\to\infty} \frac{\ln(x)}{x^2} = \frac{\infty}{\infty},$$

so we may apply L'Hospital's Rule. Thus, we have

$$\lim_{x\to\infty} \frac{\ln(x)}{x^2} = \lim_{x\to\infty} \frac{\frac{\mathrm{d}}{\mathrm{dx}}\ln(x)}{\frac{\mathrm{d}}{\mathrm{dx}}x^2} = \lim_{x\to\infty} \frac{\frac{1}{x}}{2x} = \lim_{x\to\infty} \frac{1}{2x^2} = 0.$$

So L'Hospital's Rule has succeeded, and we are very happy.