Pseudospherical Surfaces and Soliton Equations

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Abstract

In this thesis, we explore the applications of pseudospherical surface theory to the theory of soliton equations. We begin by surveying the classical differential geometry of curves and surfaces. Subsequently, we specialize to the case of pseudospherical surfaces, and discuss at length their connection with local solutions to the sine-Gordon equation, including a classification of pseudospherical surfaces of revolution, and an analytical reformulation of a celebrated theorem of Hilbert. We then relate transformations of pseudospherical surfaces to analytical machinery which allows one to produce a wealth of soliton solutions to the sine-Gordon equation. After a brief introduction to the theory of soliton equations, we discuss the manner in which a class of soliton equations beyond sine-Gordon describe pseudospherical surfaces.

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Contents

1 Introduction

The goal of this thesis is to explore the connection between a certain class of surfaces, called pseudo-spherical surfaces, and the sine-Gordon equation (sG):

$$\phi_{xt} = \sin\phi,$$

and to ultimately suggest the deep relationship between classical surface theory and the theory of soliton equations. Note that here, and in the remainder of this work, subscripts shall denote differentiation.

The first recorded mention of what we now call a soliton appeared in the writings of the Scottish engineer John Scott Russell in 1845. He noticed that, when a horse-drawn boat moving through a canal abruptly stopped, there amassed a wave of water which moved through the canal for one or two miles, maintaining its shape save a very gradual diminution in size (see chapter zero of [3] for a more extensive review of this history). Scientists debated for decades whether such stable waves could exist in shallow canals. Although first written down by Boussinesq in 1877, the equation

$$u_t - u_{xxx} - 6uu_x = 0$$

was rediscovered by Korteweg and DeVries in 1895 as a model for the dynamics of water in shallow canals, and is so named the KdV equation. Solitons were formally discovered in 1965 by Zabusky and Kruskal [15], who numerically found such solutions to KdV as Russell wrote about one century before. In particular, they noticed solutions consisting of localized wave packets which moved at constant velocity with no change in form, going mostly unharmed upon interaction with each other. This result was quite surprising, given that the KdV equation accounts for dispersive effects.

Two years later, Gardner, Greene, Kruskal, and Miura introduced the inverse scattering transform as a method for solving KdV by solving an associated linear problem [8]. From this method arose many natural consequences for the character of KdV: a formal interpretation of soliton solutions, so-called Bäcklund Transformations, to be discussed in detail later on, and an infinite number of conserved quantities, which we will not discuss in this thesis. This also gave a method for producing soliton solutions explicitly. Since then, many nonlinear PDEs with abundant applications in mathematical physics have been found to be solvable by some form of the inverse scattering transform, and these equations likewise exhibit soliton solutions, Bäcklund Transformations, and infinitely many conversed quantities, so are termed **soliton equations**. In the literature, these PDEs are often also referred to as integrable or Hamiltonian systems, and they can indeed be understood as infinite-dimensional Hamiltonian systems, but we shall exclusively use the term 'soliton equation.'

As the reader may have guessed, sG is a soliton equation, so exhibits soliton solutions, Bäcklund Transformations, and an associated linear problem which one can apply the inverse scattering transform to. Our goal is to produce all of this machinery without using any results from soliton theory. This may seem strange, but this is actually what happened historically: the associated linear problem essentially comes from the fundamental equations for surfaces, which were even partially known to Gauss, and the geometers of the late nineteenth century found that pseudospherical surfaces admit certain transformations which naturally correspond to transformations of sG solutions. In fact, the Bäcklund Transformations enjoyed by all soliton equations get their name from a transformation of pseudospherical surfaces discovered by the Swedish mathematician Albert Victor Bäcklund. So, the geometers of the time were unknowingly laying foundations for the theory of soliton equations decades before the birth of the subject!

Now that we have our sights set, let us give a brief overview of the content to be discussed in this thesis. We will begin by reviewing the classical theory of curves and surfaces in $\mathbb{R}^3$, assuming from the reader a reasonable familiarity with multivariable calculus, linear algebra, and general topology. Then, we will discuss pseudospherical surfaces and their connection with sG. We will linger on this relationship, deriving the associated linear problem for sG which is amenable to inverse scattering, exploring pseudosherical surfaces of revolution and their sG solutions, as well as recasting a theorem of Hilbert in terms of sG. We will then move on to the transformations of pseudospherical surfaces, which ultimately allow one to produce a wealth of solutions to sG by mostly algebraic means. We then tersely discuss the inverse scattering method for solving the KdV equation, and suggest how the method is generalized to become applicable to many other soliton equations. Lastly, we state the relationship between the inverse scattering method and the classical theory of pseudospherical surfaces which we will have developed, mentioning also the other, more direct relationships which surface theory has with various soliton equations. Keep in mind that our discussion of inverse scattering will be quite informal; the focus of this thesis is geometry, and we only wish to convey the basic ideas of soliton theory. The presentation of this subject only stands to show that the simple and beautiful geometry which we discuss has incredibly powerful applications in nonlinear PDE theory and mathematical physics.

2 The Classical Theory of Curves and Surfaces

2.1 Curves in $\mathbb{R}^3$

Before discussing the theory of surfaces, we must understand their one-dimensional analogue, namely curves. The basis of surface theory, and Riemannian geometry for that matter, lies in the description of the behavior of curves on a surface (or manifold). In what follows, we will use **differentiable** to mean that a function is as many times differentiable as we require, and **smooth** to mean that a function has derivatives of all orders.

Definition 2.1.1. *A* ***differentiable curve*** *is a differentiable map* $\alpha : I \to \mathbb{R}^3$, *where* I *is a connected subset of* $\mathbb{R}$.

Note that there is a distinction between a curve and its image: a curve $\alpha : I \to \mathbb{R}^3$ is, by definition, a *map*, while its image $\alpha(I)$ is called the **trace** of the curve. There can be many distinct curves which share same trace, and we will see why it is important to distinguish them.

For $\alpha : I \to \mathbb{R}^3$ a differentiable curve and $t \in I$, the derivative $\alpha'(t)$ can be viewed as a vector based at $\alpha(t)$ describing the velocity and direction in which a curve is traveling at t; in other words, we can view $\alpha' : I \to \mathbb{R}^3$ as a tangent vector field along the curve. If $\alpha'(t) = 0$ for some t, then there is no tangent line to the curve at t, and we cannot understand the curve's behavior through its tangent vector field, so we will dispense with this case by considering the following refinement of focus:

Definition 2.1.2. *A differentiable curve $\alpha : I \to \mathbb{R}^3$ is **regular** if $\alpha'(t) \neq 0$ for all $t \in I$.*

This allows us to make another central definition:

Definition 2.1.3. *For $\alpha : I \to \mathbb{R}^3$ a regular curve, and $t_0 \in I$, the arc length of α from t_0 is defined to be*

$$s(t) = \int_{t_0}^{t} |\alpha'(t)| dt.$$

Here we are using the usual Euclidean norm for vectors in $\mathbb{R}^3$. Since $\alpha'(t)$ is nonzero, we have that the function $s(t)$ is differentiable, and $s'(t) = |\alpha'(t)|$. It is clear that the parameter t measures the arc length of the curve from some initial point t_0 if and only if $|\alpha'(t)| \equiv 1$; that is, our curve is moving at constant unit speed. We will say that such a curve is **parametrized by arc length.** From now on, we will consider only those regular curves which are parametrized by arc length. We can always reparametrize a curve so that this is so, with the caveat that the domain of definition of the curve may change in the process.

2.1.1 The Frenet-Serret Formulas

We will now see that the geometry of a sufficiently nice curve in $\mathbb{R}^3$ is described completely by two smooth functions, namely its curvature and torsion. To develop this theory, we will gives names to the various actors involved, beginning with the following:

Definition 2.1.4. *For $\alpha : I \to \mathbb{R}^3$ a regular curve parametrized by arc length, we define $t(s) = \alpha'(s)$ to be the **tangent vector field** to the curve.*

If we take another derivative, we see that $\alpha'' : I \to \mathbb{R}^3$ represents the curves acceleration; that is, how quickly and in which direction the tangent vector is changing as we move along the curve α.

Definition 2.1.5. *For a regular curve $\alpha : I \to \mathbb{R}^3$ which is parametrized by arc length, we define $\kappa(s) = |\alpha''(s)|$ to be the **curvature** of α at s.*

The curvature of a curve has a more directly geometric interpretation when $\kappa \neq 0$. By placing a circle of radius $1/\kappa(s)$ in $\mathbb{R}^3$ such that its center point is $\alpha(s) + \alpha''(s)$, and the circle lies in the plane spanned by $t(s)$ and $\alpha''(s)$, we see that the the point $\alpha(s)$ lies on the circle, and the tangent to the circle at $\alpha(s)$ lies in the line spanned by $t(s)$. Thus, the curvature κ gives some intuitive notion of the *radius* *of curvature* of a curve at any point, namely the radius of a circle which locally resembles the curve. If you like, this works equally well for $\kappa = 0$, since the curve is α is tangent to a line there, which can be thought of as a circle of infinite radius.

A quick computation shows that the curvature of a curve is everywhere zero if and only if the curve is a straight line. At points where $\kappa(s) \neq 0$, there is yet more interesting geometry going on.

Definition 2.1.6. *For $\alpha : I \to \mathbb{R}^3$ a regular curve parametrized by arc length, and $s \in I$ a point at which $\kappa(s) \neq 0$, the* ***normal vector*** *$n(s)$ to the curve α at s is defined by the formula*

$$\alpha''(s) = \kappa(s)n(s)$$

.

Since $t(s) \cdot t(s) = 1$, differentiating gives that $\alpha''(s) \cdot t(s) = 0$, so that $n(s) \cdot t(s) = 0$ for all s. Thus, the normal vector to a curve is, in fact, normal to its tangent vector at all points for which $\kappa \neq 0$. The span of the vectors $t(s)$ and $n(s)$ uniquely define a plane in $\mathbb{R}^3$ containing the point $\alpha(s)$, which we call the **osculating plane** of the curve α at s. We can likewise consider this plane to be defined by the unit vector $t(s) \wedge n(s)$, where $\wedge$ is the standard vector (or cross) product in $\mathbb{R}^3$.

Definition 2.1.7. *For $\alpha : I \to \mathbb{R}^3$ a regular curve parametrized by arc length, and $s \in I$ a point at which $\kappa(s) \neq 0$, the* ***binormal vector*** *$b(s)$ to the curve α is given by*

$$b(s) = t(s) \wedge n(s).$$

We have now constructed a very natural orthonormal frame for a regular curve with non-vanishing curvature, namely the vector fields t, n, and b, which comprise the **Frenet frame** of the curve. From now on we will consider only those curves for which $\kappa \neq 0$ holds everywhere. From our definition of the binormal b, we see that its derivative $b'(s)$ measures the rate at which the curve bends away from its osculating plane at s. Since $|b(s)| \equiv 1$, its tangent vector $b'(s)$ is normal to $b(s)$. Moreover, since the vector product obeys a Leibnizian rule for differentiation, we have that

$$b'(s) = t'(s) \wedge n(s) + t(s) \wedge n'(s).$$

Since $t'(s)$ is parallel to $n(s)$, the first term vanishes, and we see that $b'(s)$ is normal to $t(s)$, which implies that $b'(s)$ is parallel to the normal vector $n(s)$. Thus, we can make another definition:

Definition 2.1.8. *For $\alpha : I \to \mathbb{R}^3$ a regular curve parametrized by arc length such that $\kappa(s) \neq 0$ for all $s \in I$, we define the* ***torsion*** *$\tau(s)$ of the curve α by the formula*

$$\tau(s) = |b'(s)|.$$

As with our definition of the curvature κ, we see that $b'(s) = \tau(s)n(s)$. We will call a curve whose trace lies entirely in some plane $P \subset \mathbb{R}^3$ a plane curve. With this definition, we can immediately state the following:

Proposition 2.1.1. *Let $\alpha : I \to \mathbb{R}^3$ a regular curve parametrized by arc length such that $\kappa(s) \neq 0$ for all $s \in I$. Then α is a plane curve if and only if $\tau \equiv 0$.*

Proof. Supposing that α is a plane curve, we have that the curve lies entirely in the plane spanned by $t(s)$ and $n(s)$ for any $s \in I$, which is exactly the osculating plane of α at some s. Since the osculating plane $\alpha(s)$ is independent of the point $s \in I$, we have that the binormal b forms a parallel vector field along alpha, so that $\tau \equiv 0$.

Conversely, if $\tau \equiv 0$, then the vector field b is constant on I, and in particular the plan spanned by $t(s)$ and $n(s)$ is independent of the point $s \in I$. Thus, the curve lies entirely in the plane spanned by $t(s)$ and $n(s)$ for any $s \in I$. □

Let us collect what we know about our curve α. We have the Frenet frame $\{t, n, b\}$, and by definition we know that $t' = \kappa n$ and $b' = \tau n$. Since $n = b \wedge t$, differentiating gives $n' = -\kappa t - \tau b$. So, we have the relations

$$t = \kappa n$$

$$n' = -\kappa t - \tau b$$

$$b' = \tau n.$$

These are know as the **Frenet-Serret formulas**. We have expressed the derivatives of our framing vector fields with respect to the curve's parameter in terms of the vector fields themselves, as well as the curvature and torsion functions. Since curvature measures a curves deviation from a straight line, and torsion measures its deviation from a plane, and we have no more dimensions in which the curve may 'bend,' it stands to reason that these curvature and torsion functions completely describe the curve in some sense. This is indeed the case:

Theorem. *(Fundamental Theorem of Curves) Given $\kappa, \tau : I \to \mathbb{R}$ differentiable functions such that $\kappa > 0$, there exists $\alpha : I \to \mathbb{R}^3$, a regular curve parametrized by arc length such that κ and τ are the curvature and torsion of the curve, respectively, and any other curve $\beta : I \to \mathbb{R}^3$ having the same curvature and torsion as α differs from α by rigid motion; that is, rotation and translation.*

The theorem follows from an existence and uniqueness theorem in ODE theory, see pages 315-317 of [4] and the references therein.

2.2 Regular Surfaces

We now move up one dimension and focus our attention to regular surfaces in $\mathbb{R}^3$. Although perhaps not transparent on a first reading, the definition is motivated by the desire to have some notion of calculus on geometric objects which sit nicely in $\mathbb{R}^3$. Although Euler made progress in understanding the local behavior of surfaces in 1760, it was largely Gauss who began the study of surfaces in the manner which we will discuss, and it was his work which motivated the development of the more general theory of Riemannian geometry, which Riemann set forth while working under Gauss.

Definition 2.2.1. *A subset $S \subset \mathbb{R}^3$ is a* ***regular surface*** *if, for each $p \in S$, there exists a neighborhood $V \subset S$ of p and a differentiable homeomorphism $\boldsymbol{x} : U \to V$, where U is an open subset of $\mathbb{R}^2$, such that the differential $d\boldsymbol{x}_q : \mathbb{R}^2 \to \mathbb{R}^3$ is injective for all $q \in U$.*

Note that we are using the subspace topology of $S \subset \mathbb{R}^3$. In particular, a set $V \subset S$ is open in S if and only if there is some open set $U \subset \mathbb{R}^3$ such that $U \cap S = V$. Thus, we require a regular surface to be locally homeomorphic to $\mathbb{R}^2$ in such a way that the homeomorphism is differentiable in the sense of standard multivariable calculus. Note that from hereon, we will always assume the domain of a parametrization to be open, and more generally any sets named U or V are assumed to be open. The maps $\mathbf{x} : U \subset \mathbb{R}^2 \to S$ from the definition are referred to as **parametrizations** or **coordinate neighborhoods** of the surface. The regularity condition requiring that the differential of each parametrization is everywhere injective is really just the two-dimensional analogue of the regularity condition on curves, namely that $\alpha' \neq 0$. We will see that this condition allows us to associate to each point on a regular surface a tangent plane, which is essential for the study of surfaces.

Roughly speaking, a regular surface is a collection of open sets in the plane stitched together in a coherent manner. As it stands though, we see that there may be many coordinate expressions for the same point on a surface, so that it is important to understand how we may pass from between parametrizations of our surface.

Proposition 2.2.1. *(Change of Parameters) For a regular surface S, a point $p \in S$, and parametrizations $\boldsymbol{x} : U \subset \mathbb{R}^2 \to S$ and $\boldsymbol{y} : V \subset \mathbb{R}^2 \to S$ such that $p \in \boldsymbol{x}(U) \cap \boldsymbol{y}(V) = W$, the change of parameter map $\boldsymbol{x}^{-1} \circ \boldsymbol{y} : \boldsymbol{y}^{-1}(W) \to \boldsymbol{x}^{-1}(W)$ is differentiable with differentiable inverse.*

Thus, we can describe a reasonable notion of differentiability for a real-valued function $f : S \to \mathbb{R}$, where S is a regular surface. In particular, we will say that this map is differentiable at $p \in S$ if, for any $\mathbf{x} : U \subset \mathbb{R}^2 \to S$ a parametrization about p, the composition $f \circ \mathbf{x} : U \to \mathbb{R}$ is differentiable at $\mathbf{x}^{-1}(p)$. We see by change of parameters that differentiability of f at p does not depend on the chosen parametrization about p, since choosing a different parametrization amounts to composition by a differentiable map. We can then also state the following:

Definition 2.2.2. *Let $\phi : V_1 \subset S_1 \to S_2$, be a map from an open subset V_1 of the regular surface S_1 to the regular surface S_2. Moreover, for some $p \in V_1$, let $\boldsymbol{x}_1 : U_1 \subset \mathbb{R}^2 \to S_1$ be a parametrization about p and $\boldsymbol{x}_2 : U_2 \subset \mathbb{R}^2 \to S_2$ a parametrization about $\phi(p)$ such that $\phi(\boldsymbol{x}_1(U_1)) \subset \boldsymbol{x}_2(U_2)$. Then we say that the map ϕ is differentiable at p if the map*

$$\boldsymbol{x}_2^{-1} \circ \phi \circ \boldsymbol{x}_1 : U_1 \to U_2$$

is differentiable at $\boldsymbol{x}_1^{-1}(p)$.

Once again, this definition does not rely on the parametrizations chosen. This allows us to establish a notion of isomorphism for regular surfaces: a **diffeomorphism** $\phi : S_1 \to S_2$ of regular surfaces is a differentiable map with differentiable inverse. Surfaces which admit a diffeomorphism are called **diffeomorphic**. Although it is not a trivial statement, two surfaces are diffeomorphic if and only if they are homeomorphic, so this notion can be thought of as properly topological despite involving the notion of differentiability. We will later see the more discerning notion of isometry which distinguishes surfaces beyond their topology.

2.2.1 The Tangent Plane

We will now develop a better understanding of regular surfaces and maps between them by introducing the notion of the tangent plane.

Definition 2.2.3. *For S a regular surface and $p \in S$, a tangent vector to S at p is the tangent vector $\alpha'(0)$ of a differentiable curve $\alpha : (-\epsilon, \epsilon) \to S$ with $\alpha(0) = p$.*

We now see how the regularity condition of our parametrizations is crucial for our study of the geometry of surface.

Proposition 2.2.2. *For $\boldsymbol{x} : U \subset \mathbb{R}^2 \to S$ a parametrization of the regular surface S, and $q \in U$ such that $\boldsymbol{x}(q) = p$, the two dimensional vector space*

$$d\boldsymbol{x}_q(\mathbb{R}^2) \subset \mathbb{R}^3$$

is the set of tangent vectors to S at p.

Note that $d\mathbf{x}_q(\mathbb{R}^2)$ is, in general, an affine subspace of $\mathbb{R}^3$, meaning that it need not contain the origin of $\mathbb{R}^3$. However, it still has the structure of a vector space in its own right. Thus, we can identify the **tangent plane** of S at p, referred to as T_pS, with the space $d\mathbf{x}_q(\mathbb{R}^2)$, since it does not depend on our choice of parametrization by the proposition. Note that our choice of parametrization gives us a basis for the tangent plane T_pS, namely the vectors $(\partial\mathbf{x}/\partial u)(q) = \mathbf{x}_u$ and $(\partial\mathbf{x}/\partial v)(q) = \mathbf{x}_v$, referred to as the basis associated to $\mathbf{x}$. In particular, any vector in T_pS has a unique expression as a linear combination of $\mathbf{x}_u(q)$ and $\mathbf{x}_v(q)$. We are now prepared to define a central notion of surface theory, namely the differential of a map, which provides a pointwise linear representation for a differentiable map by describing its associated action on the tangent spaces of the surfaces.

Definition 2.2.4. *Let $\phi : S_1 \to S_2$ be a differentiable map of regular surfaces. For each $p \in S_1$, and each $v \in T_pS_1$, choose a differentiable curve $\alpha : (-\epsilon, \epsilon) \to S_1$ such that $\alpha(0) = p$ and $\alpha'(0) = v$. Letting $\beta : (-\epsilon, \epsilon) \to S_2$ be the differentiable curve defined by $\beta = \phi \circ \alpha$, the map $d\phi_p : T_pS_1 \to T_{\phi(p)}S_2$ given by $d\phi_p(v) = \beta'(0)$ is called the **differential** of ϕ at p.*

The differential of a map is a linear map on each tangent plane which does not depend on the choice of curve α. Indeed, in a particular parametrization $\mathbf{x}(u, v) : U \subset \mathbb{R}^2 \to S_1$, our curve α has coordinate expression $\alpha(t) = (u(t), v(t))$, and the function ϕ has coordinate expression $\phi(u, v) = (\phi_1(u, v), \phi_2(u, v))$. Thus, for $\alpha'(0) = (u'(0), v'(0))$, we obtain the expression

$$d\phi_p(v) = \begin{pmatrix} \partial\phi_1/\partial u & \partial\phi_1/\partial v \\ \partial\phi_2/\partial u & \partial\phi_2/\partial v \end{pmatrix} \begin{pmatrix} u'(0) \\ v'(0) \end{pmatrix}.$$

Since the differential only depends on the tangent vector $\alpha'(0)$, the choice of curve does not affect the value of $d\phi_p(v)$. There is, unsuprisingly, a nice relationship between the differential of a map and the character of the differentiable map itself. To state this relationship, we first note that a map $\phi : S_1 \to S_2$ is called a **local diffeomorphism** at $p \in S_1$ if there exists a neighborhood $U \subset S_1$ of p such that the restriction $\phi|_U : U \to S_2$ is a diffeomorphism onto its image. We can then say the following

Proposition 2.2.3. *If $\phi : S_1 \to S_2$ is a differentiable map between regular surfaces, and for some $p \in S_1$, the differential $d\phi_p$ is an isomorphism of vector spaces, then ϕ is a local diffeomorphism at p.*

This is simply the inverse function theorem of multivariable calculus stated for surfaces.

2.2.2 The First Fundamental Form

This section begins the study of differential geometry proper for regular surfaces, in that we will begin to discuss the machinery associated to performing measurements on regular surfaces. In the previous section, we associated to any point p of a regular surface S the tangent plane T_pS sitting in $\mathbb{R}^3$. We can make T_pS into an inner product space by simply using the inner product induced by that of $\mathbb{R}^3$. We will denote this inner product by $\langle\ ,\ \rangle_p$, or sometimes simply $\langle\ ,\ \rangle$ when the point of evaluation is clear. We can then introduce the first fundamental form, which will allow us to measure lengths, angles, and areas on regular surfaces.

Definition 2.2.5. *For S a regular surface and $p \in S$, define the* ***first fundamental form*** *of S at p to be the quadratic form $I_p : T_pS \to \mathbb{R}$ given by the formula*

$$I_p(v) = \langle v, v\rangle_p.$$

Let $\mathbf{x}(u, v) : U \subset \mathbb{R}^2 \to S$ be a parametrization for our surface. Choosing a differentiable curve $\alpha : (-\epsilon, \epsilon) \to S \cap \mathbf{x}(U)$, we can write the curve α in coordinates as $\alpha(t) = (u(t), v(t))$, and thus we can write the first fundamental form at points $\alpha(t) \in S$ as

$$I_p(\alpha') = E(u')^2 + 2Fu'v' + G(v')^2,$$

where we define

$$E(u, v) = \langle \mathbf{x}_u, \mathbf{x}_u\rangle_{\alpha(t)}$$

$$F(u, v) = \langle \mathbf{x}_u, \mathbf{x}_v\rangle_{\alpha(t)}$$

$$G(u, v) = \langle \mathbf{x}_v, \mathbf{x}_v\rangle_{\alpha(t)}$$

to be the coefficients of the first fundamental form at T_pS in the basis associated to $\mathbf{x}$. Note that the parameter t was only removed to avoid notational clutter. We thus obtain three differentiable function $E, F, G : U \to \mathbb{R}$ which tell us concretely how to compute inner products in T_pS whenever $p \in \mathbf{x}(U)$. We can then define the length of curves on a surface intrinsically; that is, without direct reference to the ambient Euclidean space. Assuming that $\alpha : I \to S$ is a smooth curve on S, we define the arc length from $t_0 \in I$ to be

$$s(T) = \int_{t_0}^{T} |\alpha'(t)| dt = \int_{t_0}^{T} \sqrt{I_{\alpha(t)}(\alpha'(t))}\, dt.$$

Additionally, for curves $\alpha, \beta : I \to S$ such that $p = \alpha(t_0) = \beta(t_0)$, we can define the angle ϕ between the curves at their intersection to be given by

$$\cos\phi = \frac{\langle \alpha'(t_0), \beta'(t_0)\rangle_p}{|\alpha'(t_0)||\beta'(t_0)|}.$$

For coordinate curves in a parametrization, we see that

$$\cos\phi = \frac{\langle \mathbf{x}_u, \mathbf{x}_v\rangle_p}{|\mathbf{x}_u||\mathbf{x}_v|} = \frac{F}{\sqrt{EG}}.$$

Thus, the coordinate curves of a parametrization are orthogonal at some $p \in S$ if and only if $F(p) = 0$. Finally, computations of area can be done according to the following:

Definition 2.2.6. *For $R \subset S$ a bounded region contained in the image of a parametrization $\boldsymbol{x} : U \subset \mathbb{R}^2 \to S$, we define*

$$\iint_{\boldsymbol{x}^{-1}(R)} |\boldsymbol{x}_u \wedge \boldsymbol{x}_v| du dv$$

to be the area of the region R.

This does not depend on the parametrization because of cancelation due to the Jacobian associated to the change of parameters. We note also that we have the expression

$$|\mathbf{x}_u \wedge \mathbf{x}_v| = \sqrt{EG - F^2}$$

in terms of fundamental form coefficients, and by definition of the first fundamental form, we have that $EG - F^2 > 0$, since it defines an inner product on S which is, of course, positive-definite.. With this machinery at hand, one can calculate the area of a host of surfaces by computing its first fundamental form coefficients in particular parametrizations and performing integration, which is usually not so difficult.

2.2.3 The Gauss Map and the Second Fundamental Form

We now explore more deeply the geometry of surfaces by making use of the ambient Euclidean space. To begin, notice that for any parametrization of a regular surface $\mathbf{x} : U \subset \mathbb{R}^2 \to S$, we can define the unit normal vector field $N : \mathbf{x}(U) \to S^2$ by the formula

$$N(p) = \frac{\mathbf{x}_u \wedge \mathbf{x}_v}{|\mathbf{x}_u \wedge \mathbf{x}_v|}(\mathbf{x}^{-1}(p)).$$

We can always do this locally, but we cannot necessarily always extend this to a differentiable field of unit normal vectors on the entire surface, the canonical counterexample being the Möbius band. This calls for the following definition:

Definition 2.2.7. *A regular surface S is **orientable** if there exists a differentiable unit vector field $N : S \to S^2$ such that $N(p)$ is normal to T_pS for all $p \in S$.*

This is equivalent to the requirement that there exists a family of parametrizations and domains $\{(\mathbf{x}_\alpha, U_\alpha)\}_{\alpha \in J}$ such that $\bigcup_{\alpha \in J} \mathbf{x}_\alpha(U_\alpha) = S$, and the differential of the change of parameters $\mathbf{x}_\beta^{-1} \circ \mathbf{x}_\alpha$ has positive determinant for all $\alpha, \beta \in J$ such that $U_\alpha \cap U_\beta \neq \emptyset$, but we will not use this interpretation in what follows.

For a regular orientable surface S, the differentiable unit normal vector field $N : S \to S^2$ is called the **Gauss map**. The differential of the Gauss map at some $p \in S$ is a linear map $dN_p : T_pS \to T_{N(p)}S^2$, and a little thought justifies that these tangent planes can be regarded as the same object, so that dN_p can be regarded as a linear endomorphism on T_pS. Indeed, if we regard each of these tangent planes as affine subspaces of $\mathbb{R}^3$, where S^2 is embedded in $\mathbb{R}^3$ in the standard way, we see that they are parallel.

If we have some curve on our surface S, its composition under N, sometimes referred to as its **spherical image**, is the restriction of the unit normal vector field to the curve. Moreover, the differential of this map at some point p is the vector in T_pS which describes how the unit normal vector is changing at p as one moves along the curve. Thus, the differential of the Gauss map captures how a surface is bending away from its tangent plane as one moves in some specified direction away from an initial point p. For this reason, the map dN_p is the natural place to explore some notion of curvature for surfaces. We first remark an important property of the Gauss map:

Definition 2.2.8. *The differential $dN_p : T_pS \to T_pS$ of the Gauss map is self-adjoint.*

We will include the proof, mostly to recall the definition of a self-adjoint linear operator.

Proof. Let $\mathbf{x}(u, v) : U \subset \mathbb{R}^2 \to S$ be a parametrization of the oriented surface S about some point $p \in S$, and set $dN_p(\mathbf{x}_u) = N_u$ and $dN_p(\mathbf{x}_v) = N_v$. By differentiating the identities $\langle N, \mathbf{x}_u \rangle = 0$ and $\langle N, \mathbf{x}_v \rangle = 0$, and using the fact that $\mathbf{x}_{uv} = \mathbf{x}_{vu}$, we obtain that

$$\langle dN_p(\mathbf{x}_u), \mathbf{x}_v \rangle = \langle N_u, \mathbf{x}_v \rangle = -\langle N, \mathbf{x}_{uv} \rangle = \langle N_v, \mathbf{x}_u \rangle = \langle \mathbf{x}_u, dN_p(\mathbf{x}_v) \rangle.$$

Since $\{\mathbf{x}_u, \mathbf{x}_v\}$ is a basis for T_pS, self-adjointness in the general case follows. □

The fact that dN_p is self-adjoint allows us to define a symmetric bilinear form on T_pS sending a vectors $v, w \in T_pS$ to the inner product $\langle dN_p(v), w \rangle_p = \langle v, dN_p(w) \rangle_p$, , which justifies the following:

Definition 2.2.9. *The quadratic form $II_p : T_pS \to \mathbb{R}$ defined by*

$$II_p(v) = -\langle dN_p(v), v \rangle_p$$

*is called the **second fundamental form** of S at p.*

Likewise for the first fundamental form, we can obtain coordinate expressions for the second fundamental form. Let $\mathbf{x} : U \subset \mathbb{R}^2 \to S$ be a parametrization for the regular, oriented surface S, and $\alpha : (-\epsilon, \epsilon) \to S \cap \mathbf{x}(U)$ a differentiable curve with coordinate expression be $\alpha(t) = (u(t), v(t))$. Then we have that

$$II_p(\alpha') = e(u')^2 + 2fu'v' + g(v')^2,$$

where

$$e = -\langle N_u, \mathbf{x}_u \rangle$$
$$f = -\langle N_v, \mathbf{x}_u \rangle = -\langle N_u, \mathbf{x}_v \rangle$$
$$g = -\langle N_v, \mathbf{x}_v \rangle$$

are the coefficients of the second fundamental form expressed in the basis associated to the parametrization $\mathbf{x}$. Again, we obtain three differentiable functions $e, f, g : U \to \mathbb{R}$ which tell us how to compute the differential of the Gauss map at any point of $\mathbf{x}(U)$.

We will be able to understand the second fundamental form in terms of the behavior of curves on a surface passing through a point via the concept of normal curvature, first studied by Euler.

Definition 2.2.10. *For a regular curve $\alpha : (-\epsilon, \epsilon) \to S$ on the regular, oriented surface S such that $\alpha(0) = p \in S$, let κ be the curvature of α at p, and let $\cos\theta = \langle n, N \rangle_p$, where n is the normal vector to α at p, and N is the normal vector to S at p. Then the value*

$$k_n = \kappa \cos\theta$$

*is the **normal curvature** of α at p.*

Intuitively, the normal curvature is the length of the projection of the acceleration vector α'' onto the normal vector N to the surface; that is, it measures the extent to which α is forced to curve in order to remain on the surface. Note that the value of normal curvature does not depend on the orientation or parametrization of the curve, so we could have just as well defined the normal curvature according to the trace of a curve, rather than the curve itself. We then have the following relationship between the second fundamental form and normal curvature:

Proposition 2.2.4. *Let $\alpha : (-\epsilon, \epsilon) \to S$ be a regular curve parametrized by arc length on the regular surface, oriented surface S such that $\alpha(0) = p \in S$ and let n be the normal vector of α at $\alpha(0)$. Then*

$$II_p(\alpha'(0)) = k_n(p).$$

Proof. Since $\alpha'(s) \in T_{\alpha(s)}S$ for all $s \in (-\epsilon, \epsilon)$, we have that $\langle N(s), \alpha'(s) \rangle_{\alpha(s)} = 0$, where $N(s)$ is the spherical image α. Thus, we have that

$$II_p(\alpha'(0)) = -\langle dN_p(\alpha'(0)), \alpha'(0) \rangle_p = -\langle N'(0), \alpha'(0) \rangle_p = \langle N(0), \alpha''(0) \rangle_p,$$

where the last identity is obtained by differentiating the expression remarked to be zero above. Well, the last expression is $\langle N(0), \kappa n(0) \rangle_p$, which is precisely the normal curvature of α at p, namely $k_n(p)$. □

Thus, the normal curvature at p of a curve passing through p is determined entirely by its tangent vector there. Moreover, since we are only considering unit tangent vectors, we can regard the possible values of normal curvature of curves passing through $p \in S$ as the values of the second fundamental form restricted to the unit circle $S^1 \subset T_pS$. Since the second fundamental form is the quadratic form associated to a self-adjoint linear map, a theorem from linear algebra tells us that there exists an orthonormal basis $\{v_1, v_2\}$ for T_pS such that $dN_p(v_1) = -k_1v_1$ and $dN_p(v_2) = -k_2v_2$, where, assuming that $k_1 \geq k_2$, the values k_1 and k_2 are the maximum and minimum of the normal curvature at p, respectively. Thus, we may choose orthogonal unit vectors in $v_1, v_2 \in T_pS$ such that v_1 and v_2 are eigenvectors of the second fundamental form, and their eigenvalues are the maximum and minimum of the normal curvature at p.

Definition 2.2.11. *The eigenvalues k_1 and k_2 described above are called the* ***principal curvatures*** *at p, and the corresponding directions associated to the eigenvectors are the* ***principal directions*** *at p.*

This leads us to point out a distinguished set of curves on a regular, orientable surface.

Definition 2.2.12. *A regular curve $\alpha : I \to S$ on a regular, orientable surface S whose tangent vector $t(s)$ is along a principal direction for all $s \in I$ is called a* ***line of curvature*** *of S.*

Note that this definition does not depend on the choice of parametrization for the curve; the definition only insists certain behavior of the trace of the curve. We mention that the coordinate curves of a parametrization $\mathbf{x} : U \subset \mathbb{R}^2 \to S$ are lines of curvature if and only if $F \equiv f \equiv 0$. In this case, we call $\mathbf{x}$ a **lines of curvature parametrization**. We also have the following characterization of lines of curvature in terms of the Gauss map:

Proposition 2.2.5. *A regular curve $\alpha : I \to S$ is a line of curvature of S if and only if*

$$N'(t) = \lambda(t)\alpha'(t),$$

where $\lambda : I \to \mathbb{R}$ is a differentiable function, and $N(t)$ is the spherical image of $\alpha(t)$.

This follows from the fact that the tangent vector field of the curve is everywhere an eigenvector of the second fundamental form. We are now ready to introduce the two notions of curvature for regular, orientable surfaces.

Definition 2.2.13. *For S a regular, oriented surface, the* **Gaussian curvature** *K of S at $p \in S$ is given by*

$$K(p) = det\ (dN_p),$$

and the ***mean curvature*** *H of S at p is given by*

$$H(p) = -\frac{1}{2}Tr\ (dN_p).$$

These quantities do not depend on the choice of basis for T_pS, so choosing the orthonormal basis $\{v_1, v_2\}$ which diagonalizes the matrix dN_p, we see that the Gaussian and mean curvature are given in terms of the principal curvatures as

$$K = k_1 k_2 \qquad H = \frac{k_1 + k_2}{2}.$$

The sign of the Gaussian curvature is independent of the orientation chosen, and has important ramifications for the behavior of a surface, so we establish some vocabulary:

Definition 2.2.14. *A point p of the regular, oriented surface S is called* ***planar*** *if $dN_p = 0$,* ***parabolic*** *if $K(p) = 0$ and $dN_p \neq 0$,* ***hyperbolic*** *if $K(p) < 0$, and* ***elliptic*** *if $K(p) > 0$.*

For example, the points of a sphere are elliptic, the points of a plane are planar, the points of a cylinder are parabolic, and we will see plenty of hyperbolic points in our discussion of pseudospherical surfaces, although points near the center of a pringle are a nice example.

We collect one more term here which will be important in our study of pseudospherical surfaces.

Definition 2.2.15. *For S a regular, orientable surface, an* ***asymptotic direction*** *at $p \in S$ is a direction in T_pS for which the normal curvature is zero, and an* ***asymptotic curve*** *is one whose tangent vector field is everywhere along an asymptotic direction.*

Note that at elliptic points of a surface, the principal curvatures do not differ sign, so that there are no asymptotic directions. We will see later that at hyperbolic points, there are two distinct asymptotic directions which bisect the principal directions. Given some parametrization $\mathbf{x} : U \subset \mathbb{R}^2 \to S$, and a curve $\alpha : I \to S \cap \mathbf{x}(U)$ with coordinate expression $\alpha(t) = (u(t), v(t))$, the condition that α be an asymptotic curve is that $II_p(\alpha') \equiv 0$. In coordinates this reads

$$e(u')^2 + 2fu'v' + g(v')^2 = 0.$$

This tells us that the coordinate curves of a parametrization such that all points in $\mathbf{x}(U)$ are hyperbolic are asymptotic curves if and only if $e \equiv g \equiv 0$. If this is the case, we call $\mathbf{x}$ a parametrization in asymptotic coordinates, or simply an **asymptotic parametrization**. The local existence of such parametrizations at hyperbolic points of a surface is guaranteed:

Proposition 2.2.6. *For S a regular, orientable surface, and $p \in S$ a hyperbolic point, there exists an asymptotic parametrization about a neighborhood of p.*

In order to actually compute the curvature of a surface, one needs to work in coordinates. Supposing that $\mathbf{x} : U \subset \mathbb{R}^2 \to S$ is a parametrization for the regular, oriented surface S, a somewhat lengthy computation yields

$$K = \frac{eg - f^2}{EG - F^2} \qquad H = \frac{1}{2}\frac{eG - 2fF + gE}{EG - F^2},$$

so we obtain expressions in terms of the first and second fundamental forms obtained for our coordinate system.

To conclude this section, we define the more refined notion of equivalence of regular surfaces which takes into account their geometry:

Definition 2.2.16. *A diffeomorphism $\phi : S_1 \to S_2$ is an **isometry** if*

$$\langle u, v\rangle_p = \langle d\phi_p(u), d\phi_p(v)\rangle_{\phi(p)}$$

for all $p \in S_1$ and all $u, v \in T_pS_1$.

Surfaces admitting an isometry are said to be **isometric**, and the notion of local isometry is defined in the obvious way. The following proposition captures the fact that the first fundamental form is invariant under local isometry.

Proposition 2.2.7. *If parametrizations $\boldsymbol{x}_1 : U \subset \mathbb{R}^2 \to S_1$ and $\boldsymbol{x}_2 : U \subset \mathbb{R}^2 \to S_2$ are such that $E_1 \equiv E_2$, $F_1 \equiv F_2$, and $G_1 \equiv G_2$ on U, then $\boldsymbol{x}_2 \circ \boldsymbol{x}_1^{-1} : \boldsymbol{x}(U) \to S_2$ is a local isometry.*

2.2.4 The Fundamental Equations

For S a regular orientable surface, choose some orientation $N : S \to S^2$, and let $\mathbf{x} : U \subset \mathbb{R}^2 \to S$ be a parametrization which is compatible with the orientation, meaning that

$$\frac{\mathbf{x}_u \wedge \mathbf{x}_v}{|\mathbf{x}_u \wedge \mathbf{x}_v|}(\mathbf{x}^{-1}(p)) = N(p)$$

for all $p \in \mathbf{x}(U)$. Then the set of vector fields $\{\mathbf{x}_u, \mathbf{x}_v, N\}$ frames our surface, just as the Frenet trihedron frames a curve. We can then express the derivatives of such vector fields in this frame, with

$$\begin{aligned}
\mathbf{x}_{uu} &= \Gamma^1_{11}\mathbf{x}_u + \Gamma^2_{11}\mathbf{x}_v + \alpha_1 N\\
\mathbf{x}_{uv} &= \Gamma^1_{12}\mathbf{x}_u + \Gamma^2_{12}\mathbf{x}_v + \alpha_2 N\\
\mathbf{x}_{vu} &= \Gamma^1_{21}\mathbf{x}_u + \Gamma^2_{21}\mathbf{x}_v + \alpha_2' N\\
\mathbf{x}_{vv} &= \Gamma^1_{22}\mathbf{x}_u + \Gamma^2_{22}\mathbf{x}_v + \alpha_3 N\\
N_u &= a_{11}\mathbf{x}_u + a_{21}\mathbf{x}_v\\
N_v &= a_{12}\mathbf{x}_u + a_{22}\mathbf{x}_v
\end{aligned}$$

which we shall call the **Christoffel system**. The Γ^k_{ij} are the **Christoffel symbols** of S in the parametrization $\mathbf{x}$. For the reader familiar with Riemannian geometry, these are in fact the usual Christoffel symbols of the Levi-Civita connection on S, where we view S as a 2-manifold which is isometrically embedded in $\mathbb{R}^3$. We immediately see that $\Gamma^k_{12} = \Gamma^k_{21}$ from the identity $\mathbf{x}_{uv} = \mathbf{x}_{vu}$, and similarly we have $\alpha_2 = \alpha_2'$. By taking inner products with N, we see that $\alpha_1 = e$, $\alpha_2 = f$, and $\alpha_3 = g$, and taking inner products of the expressions for N_u and N_v with $\mathbf{x}_u$ and $\mathbf{x}_v$, we end up with the equation

$$-\begin{pmatrix} e & f \\ f & g \end{pmatrix} = \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix}\begin{pmatrix} E & F \\ F & G \end{pmatrix}$$

from which we can expressions for the a_{ij} coefficients in terms of the fundamental forms, although we will not have occasion to use these. As for the Christoffel symbols, a long computation gives

$$\begin{aligned}
\Gamma^1_{11} &= \frac{1}{2}\frac{E_uG - E_vG - 2FF_u}{EG - F^2} \\
\Gamma^2_{11} &= \frac{1}{2}\frac{-E_uF - E_vE + 2EF_u}{EG - F^2} \\
\Gamma^1_{12} = \Gamma^1_{21} &= \frac{1}{2}\frac{E_vG - G_uF}{EG - F^2} \\
\Gamma^2_{12} = \Gamma^2_{21} &= \frac{1}{2}\frac{G_uE - E_vF}{EG - F^2} \\
\Gamma^1_{22} &= \frac{1}{2}\frac{2F_vG - G_uG - G_vF}{EG - F^2} \\
\Gamma^2_{22} &= \frac{1}{2}\frac{G_vE + G_uF - 2FF_v}{EG - F^2}
\end{aligned} \tag{1}$$

Since the Christoffel symbols are expressible in terms of the first fundamental form coefficients only, they are intrinsic quantities in the sense that, once we define an inner product on each tangent space of our surface, we can compute the Christoffel symbols without any reference to the ambient Euclidean space in which our surface sits. Commutativity of the operators $\partial/\partial u$ and $\partial/\partial v$ gives us the identities:

$$(\mathbf{x}_{uu})_v - (\mathbf{x}_{uv})_u = 0$$

$$(\mathbf{x}_{vv})_u - (\mathbf{x}_{vu})_v = 0$$

$$N_{uv} - N_{vu} = 0,$$

which can be written in the frame $\{\mathbf{x}_u, \mathbf{x}_v, N\}$, and by linear independence the nine coefficients must all vanish. We will not include the computations, but manipulation of these expressions can lead to

$$(\Gamma^2_{12})_u - (\Gamma^2_{11})_v + \Gamma^1_{12}\Gamma^2_{11} + (\Gamma^2_{12})^2 - \Gamma^2_{11}\Gamma^2_{22} = \Gamma^1_{11}\Gamma^2_{12} = -EK,$$

which is known as the **Gauss equation**. This shows that the Gaussian curvature of our surface is an intrinsic property, which is quite striking considering that our definition of the Gaussian curvature seemed to rely heavily on the ambient space through use of the second fundamental form. In particular, if we gave some regular, orientable surface a first fundamental form induced by the Euclidean metric in $\mathbb{R}^3$, we could then completely forget that S sits in $\mathbb{R}^3$, and we would still be able to compute the Gaussian curvature of the surface. This is the content of the following theorem of Gauss, which the great mathematician was enough impressed with to call the *remarkable theorem*:

Theorem 2.2.1. *(Theorema Egregium) The Gaussian curvature of a surface is invariant under local isometries.*

Similar manipulations which gave the Gauss equation also yield the relations

$$e_v - f_u = e\Gamma^1_{12} + f(\Gamma^2_{12} - \Gamma^1_{11}) - g\Gamma^2_{11}$$

$$f_v - g_u = e\Gamma_{22}^1 + f(\Gamma_{22}^2 - \Gamma_{12}^1) - g\Gamma_{12}^2,$$

which are called the **Mainardi-Codazzi equations.** Although the journey is much more complicated, we have done something reminiscent to finding the Frenet-Serret formulas for curves: we have expressed the derivatives of the vector fields which frame our surface in terms of the vector fields themselves, as well as the fundamental form coefficients. It is far from clear whether or not the equations we have obtained are enough to characterize the surface, but the following result of Bonnet tells us that, at least locally, they are.

Theorem 2.2.2. *(Bonnet) Let $E, F, G, e, f, g : U \subset \mathbb{R}^2 \to \mathbb{R}$ be differentiable functions such that $E, G > 0$, which satisfy the Gauss and Mainardi-Codazzi equations and the relation $EF - G^2 > 0$. Then, for each $p \in U$, there exists a neighborhood $V \subset U$ of p and a differentiable embedding $\boldsymbol{x} : V \to \mathbb{R}^3$ such that the regular surface $\boldsymbol{x}(V)$ has fundamental form coefficients the given functions. Moreover, if V is connected, for any other embedding $\boldsymbol{y} : V \to \mathbb{R}^3$ satisfying the same conditions, the regular surfaces $\boldsymbol{x}(V)$ and $\boldsymbol{y}(V)$ differ only by rigid motion.*

This theorem justifies the terminology that the Gauss and Mainardi-Codazzi equations are the compatibility conditions, or fundamental equations, for a surface.

3 Pseudospherical Surfaces

3.1 Prelude

Although differential equations show up in many contexts in surface theory, there is no reason to suspect a priori that surface theory can be a useful tool for solving differential equations. However, it is largely the goal of this thesis to explain that this is the case. To begin telling this story, we turn our attention to a special class of surfaces:

Definition 3.1.1. *A regular, orientable surface S is called **pseudospherical** if $K(p) = -1$ for all $p \in S$.*

We will see that, for a suitable choice of coordinate system, the compability conditions for a pseudospherical surface boil down to one of the two equations

$$\phi_{uu} - \phi_{vv} = \sin\phi\cos\phi$$

$$\phi_{xt} = \sin\phi,$$

which connection was first set down in 1862 by Edmond Bour [2]. We have already seen the second equation, but they are indeed two different forms of sG, up to a change of coordinates and rescaling by a constant. Since we will be exploring sG in some detail, we note here that sG is known to fairly accurately describe various systems in nonlinear physics: it appears in the Frenkel-Kontorova model for crystal dislocations, and serves as a useful approximation in the study of long Josephson junctions, to name just a couple of applications.

3.2 Coordinate Systems

We have seen that surfaces admit local parametrizations in asymptotic coordinates at hyperbolic points. In the case of pseudospherical surfaces, we can actually cover our surface by parametrizations in asymptotic coordinates since every point is hyperbolic. In particular, for any p a point of the pseudospherical surface S, there exists a parametrization $\mathbf{x}(x,t) : U \subset \mathbb{R}^2 \to S$ such that $p \in \mathbf{x}(U)$, and the coordinate curves are asymptotic. In such a parametrization, the second fundamental form reads

$$II(x,t) = \lambda(x,t)dxdt,$$

where $\lambda : U \to \mathbb{R}$ is a positive, differentiable function. The parametrization $\mathbf{x}$ is given by differentiable coordinate functions, with

$$\mathbf{x}(x,t) = (f(x,t), g(x,t), h(x,t)).$$

Under the change or coordinates

$$x = u + v$$

$$t = u - v,$$

we obtain the parametrization $\mathbf{x} : V \to S$ expressed in the variables u and v defined by the coordinate functions

$$\mathbf{x}(u,v) = (f(u,v), g(u,v), h(u,v)),$$

having the same image in S as before. After this change of variables, the second fundamental form appears as

$$II(u,v) = \lambda(u,v)\left(du^2 - dv^2\right)$$

A parametrization in such coordinates (namely, where $e = -g$ and $f = 0$) will be called **isothermal-conjugate**. The constant curvature condition on S immediately yields the relation

$$\lambda = -\sqrt{EG - F^2},$$

where the choice of sign is inconsequential. The Mainardi-Codazzi equations now take the simple form

$$\lambda_u = \lambda(\Gamma^1_{22} + \Gamma^2_{12})$$

$$\lambda_v = \lambda(\Gamma^1_{12} + \Gamma^2_{11}).$$

The functions λ_u and λ_v can be expressed in terms of the first fundamental form coefficients by differentiation of the identity $\lambda = -\sqrt{EG - F^2}$, with

$$\lambda_u = \frac{EG_u + GE_u - 2FF_u}{2\lambda}$$

$$\lambda_v = \frac{EG_v + GE_v - 2FF_v}{2\lambda}.$$

As for the right-hand side of the Mainardi-Codazzi equations, we can use the expressions of the Christoffel symbols in terms of the first fundamental form coefficients and their derivatives given in section 2.2.4, with

$$\lambda(\Gamma^1_{22} + \Gamma^2_{12}) = \frac{2GF_v - GG_u - FG_v + EG_u - FE_v}{2\lambda}$$
$$\lambda(\Gamma^1_{12} + \Gamma^2_{11}) = \frac{GE_v - FG_u - E_uF - EE_v + 2EF_u}{2\lambda}.$$

Putting these identities together, some simplifications give the tidy relations

$$G(E_u + G_u - 2F_v) = -F(E_v + G_v - 2F_u)$$
$$E(E_v + G_v - 2F_u) = -F(E_u + G_u - 2F_v).$$

Letting $x = E_u + G_u - 2F_v$ and $y = E_v + G_v - 2F_u$, the relations are

$$Gx = -Fy$$
$$Ey = -Fx.$$

By multiplying these equations and moving terms to one side, we see that

$$(EG - F^2)xy = 0.$$

The quantity $EG - F^2$ certainly is nonzero in the domain of our parametrization, so the product xy must vanish everywhere. Now, assume without loss of generality that, for some $q \in V$, we have $x(q) = 0$. The Mainardi-Codazzi equations read:

$$F(q)y(q) = 0$$
$$E(q)y(q) = 0.$$

Since both E and F cannot simultaneously vanish, we have $y(q) = 0$. Thus, the quantities x and y are zero everywhere; that is,

$$E_u + G_u - 2F_v = 0$$
$$E_v + G_v - 2F_u = 0.$$

Observe that the functions

$$E = \cos^2\phi \qquad F = 0 \qquad G = \sin^2\phi$$

solve such a system, where $\phi : V \to \mathbb{R}$ is a differentiable function of u and v never equal to an integer multiple of $\pi/2$. The second fundamental form coefficients in this parametrization are readily obtained:

$$e = -\sin\phi\cos\phi \qquad f = 0 \qquad g = \sin\phi\cos\phi.$$

The isothermal-conjugate coordinate system will lend itself to computational ease when considering transformations of pseudospherical surfaces, but it will also bring us directly to the correspondence which such surfaces have with sG.

3.3 Sine-Gordon as a Compatibility Condition

We have produced a candidate for the first and second fundamental forms of isothermal-conjugate neighborhoods for an arbitrary pseudopsherical in terms of some function ϕ. However, we have not tested these functions against the Gauss equation. To this end, we compute the Christoffel symbols, which are given by

$$\Gamma^1_{11} = -\phi_u \tan\phi \qquad \Gamma^1_{12} = -\phi_v \tan\phi \qquad \Gamma^1_{22} = -\phi_u \tan\phi$$
$$\Gamma^2_{11} = \phi_v \cot\phi \qquad \Gamma^2_{12} = \phi_u \cot\phi \qquad \Gamma^2_{22} = \phi_v \cot\phi,$$

and the Gauss equation simplifies to

$$\phi_{uu} - \phi_{vv} = \sin\phi\cos\phi,$$

which is the sG expressed in so-called laboratory coordinate. We will denote this equation as **l-sG** for brevity. Passing back into asymptotic coordinates, also called light-cone coordinates, requires the coordinate change

$$x = u + v$$
$$t = u - v.$$

The Gauss equation is then

$$\phi_{xt} = \sin\phi\cos\phi,$$

multiplying through by 2 and using a trigonometric identity yields the original form of sG noted in this thesis, reading

$$(2\phi)_{xt} = \sin 2\phi,$$

which we will refer to as **c-sG**. Note the apparent conflict of notation: we call u and v laboratory coordinates, and x and t light-cone coordinates, whereas x and t are a more reasonable choice for coordinates describing space and time in a laboratory. This is because we will mostly work in laboratory coordinates in geometric constructions, so that we wish to keep the convention of a parametrization having parameters u and v. After all, these names come from the reminiscence with transformations in special relativity, and we do not wish to consider this analogy in any serious manner.

Not very surprising is the fact that the function ϕ explicitly describes the geometry of the pseudospherical surface under investigation. To elucidate this significance, we see that, in the asymptotic coordinates x and t, the first fundamental form reads

$$I(x,t) = dx^2 + 2\cos(2\phi)dxdt + dt^2.$$

Since $E \equiv G \equiv 1$, the asymptotic coordinate curves are parametrized by arc length, so that 2ϕ represents the angle between asymptotic curves at each point in the coordinate neighborhood. In general, a parametrization for which $E \equiv G \equiv 1$ is called a Chebyshev* net, so we have rather found that a

*The surname of the Russian mathematician Pafnuty Chebyshev is spelled in a wide variety of ways in the Roman alphabet. We have chosen the spelling adopted by the American Mathematical Society.

pseudospherical surface can be covered by what we shall call **asymptotic Chebyshev nets**, the meaning of such terminology being clear.

Thus, any pseudospherical surface can locally be described by l-sG solutions which avoid integer multiples of $\pi/2$ (or c-sG solutions which avoid integer multiples of π). This restriction of the range of ϕ is clear when considering that the quantity $EG - F^2$ is everywhere positive, or when taking into account the fact that the angle between distinct asymptotic directions may never be π, since this implies that the directions coincide. Likewise, one can construct a pseudospherical surface from any l-sG solution avoiding integer multiples of $\pi/2$ (or c-sG solution avoiding integer multiples of π) according to Bonnet's theorem of section 2.2.4. Note that this local representation is not unique, however. Suppose that we have a function ϕ satisfying

$$\phi_{xt} = \sin\phi$$

which locally represents the surface S, so avoids integer multiples of π and describes the angle between asymptotic curves in an asymptotic Chebyshev neighborhood. Then $\phi + 2n\pi$ for $n \in \mathbb{Z}^+$ is also a c-sG solution and decribes the same surface, as well as the function $\pi - \phi$. Addition of 2π is clearly allowed since the sine function is periodic, and the solution $\pi - \phi$ reflects the fact that one can reverse the direction of travel for one of the coordinate curves of an asymptotic Chebyshev net, so that the angle between asymptotic curves is brought from ϕ in the original asymptotic Chebyshev net to $\pi - \phi$ in the modified one.

3.4 A Linear Representation

To place the obtained correspondence in a more condensed form, we can express the Christoffel system of Section 2.2.4 as a system of 2×2 linear equations which has sG as a compatibility condition. This will also bring us to a linear system which is amenable to the inverse scattering transformation. Note that this section will make modest use of Lie theory.

Assume that we have an asymptotic Chebyshev net $\mathbf{x} : U \subset \mathbb{R}^2 \to S$, of the pseudospherical surface S, so that the fundamental forms may be expressed as

$$I = dx^2 + 2\cos\phi \, dxdt + dt^2$$

$$II = 2\sin\phi \, dxdt,$$

where $\phi : U \to (0, \pi)$ is a local c-sG solution. The Christoffel symbols of this parametrization are then

$$\Gamma^1_{11} = \phi_x \cot\phi \qquad \Gamma^1_{12} = 0 \qquad \Gamma^1_{22} = -\phi_t \csc\phi$$

$$\Gamma^2_{11} = -\phi_x \csc\phi \qquad \Gamma^2_{12} = 0 \qquad \Gamma^2_{22} = \phi_t \cot\phi.$$

Defining the orthonormal triad

$$\{\mathbf{i}, \mathbf{j}, \mathbf{k}\} = \{\mathbf{x}_x, -\mathbf{x}_x \wedge \mathbf{N}, \mathbf{N}\},$$

where $\mathbf{N}$ is the unit normal vector field on S compatible with the parametrization, we can express the Christoffel system in this frame. To give an example of how the computation is done, see that

$$\mathbf{i}_x = \mathbf{x}_{xx} = \phi_x \left(\cot\phi \mathbf{x}_x - \csc\phi \mathbf{x}_t\right).$$

Since $\mathbf{N} = (\mathbf{x}_x \wedge \mathbf{x}_t)/\sin\phi$, it follows that $\mathbf{j} = -\mathbf{x}_x \wedge N = -\cot\phi \mathbf{x}_x + \csc\phi \mathbf{x}_t$, from which we simply have that $\mathbf{i}_x = -\phi_x \mathbf{j}$. Carrying on similarly for all first derivatives, we obtain six equations which we can express in matrix form as

$$\begin{pmatrix} \mathbf{i} \\ \mathbf{j} \\ \mathbf{k} \end{pmatrix}_x = \begin{pmatrix} 0 & -\phi_x & 0 \\ \phi_x & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{i} \\ \mathbf{j} \\ \mathbf{k} \end{pmatrix}$$
$$\begin{pmatrix} \mathbf{i} \\ \mathbf{j} \\ \mathbf{k} \end{pmatrix}_t = \begin{pmatrix} 0 & 0 & \sin\phi \\ 0 & 0 & -\cos\phi \\ -\sin\phi & \cos\phi & 0 \end{pmatrix} \begin{pmatrix} \mathbf{i} \\ \mathbf{j} \\ \mathbf{k} \end{pmatrix},$$

where we consider the matrix with one column to be the 3×3 matrix whose rows are the coordinate functions for $\mathbf{i}, \mathbf{j}$, and $\mathbf{k}$. What is nice about this representation is that the Mainardi-Codazzi equations are baked into these equations: if ϕ satisfies c-sG and avoids integer multiples of π, then this system is solvable for the functions $\mathbf{i}, \mathbf{j}$, and $\mathbf{k}$, and from these one can construct the given coordinate chart $\mathbf{x}$ by the fundamental theorem of surfaces of Bonnet. To give the matrices above names, let us say that the above equations are

$$T_x = AT$$
$$T_t = BT.$$

By direct computation, we see that

$$A_t - B_x + [A, B] = \begin{pmatrix} 0 & -\phi_{xt} + \sin\phi & 0 \\ \phi_{xt} - \sin\phi & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Thus, the compatibility condition, in this case the Gauss equation expressed for our parametrization, is simply

$$A_t - B_x + [A, B] = 0,$$

which states that ϕ satisfies c-sG (note that the 0 above represents the 3×3 matrix whose entries are all zero). Since A and B are skew-symmetric, they belong to the Lie algebra $\mathfrak{so}(3)$, which has the basis

$$M_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \quad M_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \quad M_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

In particular, $A = -M_1 + \phi_x M_3$ and $B = \cos\phi M_1 + \sin\phi M_2$. Moreover, the bracket operation of $\mathfrak{so}(3)$ is the standard commutator, which gives the relations

$$[M_1, M_2] = M_3 \quad [M_2, M_3] = M_1 \quad [M_3, M_1] = M_2$$

To obtain a 2×2 linear representation from our 3×3 system, we use the isomorphism between $\mathfrak{so}(3)$ and $\mathfrak{su}(2)$, the Lie algebra consisting of complex-valued traceless 2×2 matrices which are skew-Hermitian. The Lie algebra $\mathfrak{su}(2)$ has basis

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

known as the Pauli matrices. Again, the bracket operation is given by the standard commutator, yielding the relations

$$[\sigma_1, \sigma_2] = 2i\sigma_3 \quad [\sigma_2, \sigma_3] = 2i\sigma_1 \quad [\sigma_2, \sigma_1] = 2i\sigma_2 \quad .$$

The isomorphism is clear from here; we just send M_j to $\sigma_j/2i$. Applying this isomorphism to A and B gives a 2×2 linear system

$$S_x = \tilde{A}S$$
$$S_t = \tilde{B}S,$$

where $\tilde{A}$ and $\tilde{B}$ are the images of A and B in $\mathfrak{su}(2)$, given by

$$\tilde{A} = \frac{i}{2}\begin{pmatrix} -\phi_x & 1 \\ 1 & \phi_x \end{pmatrix} \quad \tilde{B} = -\frac{i}{2}\begin{pmatrix} 0 & e^{-i\phi} \\ e^{i\phi} & 0 \end{pmatrix},$$

and S the image of T, recalling that T was the matrix whose rows were the entries of $\mathbf{i}, \mathbf{j}$, and $\mathbf{k}$. Now, introducing the gauge transformation

$$G = \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix},$$

we will define $\Phi = GS$. We then have that $\Phi_x = GS_x$ and $\Phi_t = GS_t$, from which we see that our linear system transforms to

$$\Phi_x = G\tilde{A}G^{-1}\Phi$$
$$\Phi_t = G\tilde{B}G^{-1}\Phi,$$

which, upon performing the multiplication, gives

$$\Phi_x = \frac{1}{2}\begin{pmatrix} i & -\phi_x \\ \phi_x & -i \end{pmatrix}\Phi$$

$$\Phi_t = \frac{1}{2i}\begin{pmatrix} \cos\phi & \sin\phi \\ \sin\phi & -\cos\phi \end{pmatrix}\Phi.$$

Lastly, we notice that, if we effect the transformation $\phi' = -\phi,\ u = x/\lambda,\ v = \lambda t$ for $\lambda \neq 0$, then $\phi'_{uv} = \sin\phi'$. The invariance of sG under the transformation of the x and t variables was first observed by Sophus Lie. Those familiar with modern physics will notice that this reflects the relativistic invariance of sG, which, together with its nature as a soliton equation, makes it an attractive candidate for a $(1+1)$-dimensional field theory. If we apply such a transformation to the equations above, we first see that

$$\frac{1}{\lambda}\Phi_u = \frac{1}{2}\begin{pmatrix} i & \phi'_u/\lambda \\ -\phi'_u/\lambda & -i \end{pmatrix}\Phi$$

$$\lambda\Phi_v = \frac{i}{2}\begin{pmatrix} -\cos\phi' & \sin\phi' \\ \sin\phi' & \cos\phi' \end{pmatrix}\Phi.$$

By simplifying this expression, we obtain the so-called AKNS representation for the sine-Gordon equation, namely

$$\Phi_u = \frac{1}{2}\begin{pmatrix} i\lambda & \phi'_u \\ -\phi'_u & -i\lambda \end{pmatrix}\Phi$$

$$\Phi_v = \frac{i}{2\lambda}\begin{pmatrix} -\cos\phi' & \sin\phi' \\ \sin\phi' & \cos\phi' \end{pmatrix}\Phi.$$

Note that this differs from the standard AKNS system for sG by a gauge transformation. We will mention in Section 5.2 that the first equation is the linear problem for which one can perform the inverse scattering transform and solve the initial value problems for sG. If we denote

$$L = \frac{1}{2}\begin{pmatrix} i\lambda & \phi'_u \\ -\phi'_u & -i\lambda \end{pmatrix} \qquad B = \frac{i}{2\lambda}\begin{pmatrix} -\cos\phi' & \sin\phi' \\ \sin\phi' & \cos\phi' \end{pmatrix},$$

then the equation

$$L_v - B_u + [L, B] = 0$$

still recovers c-sG.

3.5 Pseudospherical Surfaces of Revolution

We now consider a family of pseudospherical surfaces with especially simple geometry, namely rotational symmetry. We will see that these surfaces are not difficult to classify, and that they are in direct correspondence with l-sG solutions which depend non-trivially on only one variable. We begin by considering any surface of revolution, which admits a parametrization $\mathbf{x} : (0, 2\pi) \times (a, b) \to \mathbb{R}^3$ with coordinate functions given by

$$\mathbf{x}(u, v) = (f(v)\cos u, f(v)\sin u, g(v)),$$

obtained by rotating a plane curve, which we will call the **generating curve**, in the $xz-$plane about the z-axis, which is the axis of symmetry for the surface, in that rotation about this axis by any angle is an isometry. Note that, to obtain a family of charts which covers the surface, we can also include a parametrization with the same coordinate functions, but with domain $(-\pi, \pi) \times (a, b)$, which will yield the same fundamental forms as the original chart $\mathbf{x}$. The first fundamental form is then

$$I = f^2 du^2 + (f_v^2 + g_v^2) dv^2$$

and we can make the assumption that the generating curve is parametrized by arc length, which gives $G \equiv 1$. The second fundamental form is

$$II = -f g_v du^2 + (f_{vv} g_v - f_v g_{vv}) dv^2,$$

and so the Gaussian curvature of the surface is

$$K = -\frac{g_v (f_{vv} g_v - f_v g_{vv})}{f}.$$

By the assumption that the generating curve is parametrized by arc length, we obtain $f_v f_{vv} = -g_v g_{vv}$, which can be used to simplify our expression:

$$K = -\frac{g_v^2 f_{vv} - g_v g_{vv} f_v}{f} = -\frac{g_v^2 f_{vv} + f_v^2 f_{vv}}{f} = -\frac{f_{vv}}{f}.$$

Now, specializing to $K \equiv -1$, we see that $f = f_{vv}$, which has general solution

$$f(v) = \alpha e^v + \beta e^{-v}.$$

We have assumed in our computations that the generating curve never intersects the z-axis, so we can discard the trivial solution in which $\alpha = \beta = 0$. In the case when $\alpha = \beta \neq 0$, we obtain

$$f(v) = \gamma \cosh v,$$

where $\gamma = 2\alpha$. From the relation $f_v^2 + g_v^2 = 1$, we see that $g_v = \sqrt{1 - \gamma^2 \sinh^2 v}$, where the choice of sign does not alter the geometry of the resulting surface. Note that, by doing some simple algebra, we can obtain the domain for this function. Letting $a = \left| \ln\left(\sqrt{1 + 1/\gamma^2} + 1/|\gamma| \right) \right|$, it works out to be $[-a, a]$ Thus, we have that the coordinate functions completely describing the surface are

$$f(v) = \gamma \cosh v \quad g(v) = \int_0^v \sqrt{1 - \gamma^2 \sinh^2 t}\, dt,$$

where we can take the domain of the parameter v to be $(-a, a)$. We will call this a **cosh-type** surface. Integration is not so simple here; if we make the substitution $v \mapsto it$, and envoke the identity $\sinh(it) = i \sin t$, we obtain the integral

$$g(v) = -i \int_0^v \sqrt{1 + \gamma^2 \sin^2 t}\, dt,$$

which is an *incomplete elliptic integral of the second kind.* These integrals have been studied extensively and are quite interesting in their own right, but here is not the place to discuss their solutions, so we may leave the coordinate functions as above and continue with our classification. In the case when $\alpha = -\beta \neq 0$, we obtain

$$f(v) = \gamma \sinh v,$$

where $\gamma = 2\alpha$. We can again make an arbitrary choice of sign and set $g_v = \sqrt{1 - \gamma^2 \cosh^2 v}$. More algebra gives that, letting $b = \left|\ln\left(\sqrt{1/\gamma^2 - 1} + 1/|\gamma|\right)\right|$, the domain for this function is the interval $(0, b]$, and there is the additional restriction that $|\gamma| < 1$. We are restricting to the positive section of the domain because the curve crosses the z-axis, and we want to consider connected curves which do not cross this axis. Thus, we have coordinate functions

$$f(v) = \gamma \sinh v \quad g(v) = \int_0^v \sqrt{1 - \gamma^2 \cosh^2 t}\, dt,$$

with $v \in (0, b)$ and $|\gamma| \in (0, 1)$. This shall be called a **sinh-type** surface. This, again, involves an elliptic integral, so we shall leave the coordinate functions as written. We also have the simple case in which $\beta = 0$, which gives

$$f(v) = \alpha e^v \quad g(v) = \int_0^v \sqrt{1 - \alpha^2 e^{2t}}\, dt,$$

where $v \in (-\infty\ ,\ \ln(1/|\alpha|))$; to be called an **e-type** surface. The integral above can be computed explicitly, and we will do so in the next section. Note that the case in which $\alpha = 0$ and $\beta \neq 0$ produces the same surface, since the coordinate functions differ by the reparametrization $v \mapsto -v$.

Now, consider the case in which $\alpha \neq \beta \neq 0$, and α and β share the same sign. In this case, we can set $\gamma = \pm 2\sqrt{\alpha\beta}$ and $c = \frac{1}{2}\ln(\beta/\alpha)$, where the sign of gamma is determined by the sign of α and β. Some algebra yields

$$f(v) = \gamma \frac{e^{v-c} + e^{c-v}}{2} = \gamma \cosh(v - c).$$

This produces a reparametrization of a cosh-type surface, so that it is also of cosh-type. Similarly, we have the case in which the e^v and e^{-v} coefficients are not equal and differ in sign, which we can write as

$$f(v) = \alpha e^v - \beta e^{-v},$$

where α and β share the same sign. In this case, we may set $\gamma = \pm 2\sqrt{-\alpha\beta}$ and $c = \frac{1}{2}\ln(-\beta/\alpha)$, where the sign of γ is chosen to agree with the sign of α. This gives

$$f(v) = \gamma \sinh(v - c),$$

a reparametrization of a sinh-type surface, so that this surface is also of sinh-type.

To collect our results, we see that any pseudospherical surface of rotation can be described by generating curves of three types, namely:

$$f(v) = \gamma \cosh v \quad g(v) = \int_0^v \sqrt{1 - \gamma^2 \sinh^2 t}\, dt$$

$$f(v) = \gamma \sinh v \quad g(v) = \int_0^v \sqrt{1 - \gamma^2 \cosh^2 t} dt$$

$$f(v) = e^v \quad g(v) = \int_0^v \sqrt{1 - e^{2t}} dt,$$

where the relevant domains of the parameter v and possible restrictions for the value of γ have been described above. Notice that we have removed the parameter γ from the e-type surfaces: in the next section, we will see that there is only one e-type surface up to rigid motion in $\mathbb{R}^3$. The reader familiar with the notion of completeness for Riemannian manifolds will notice that none of these surfaces are complete; the domain of the parameter v for the generating curve had some restrictions in all cases, so one could find a Cauchy sequence in S with no limit in S, etc. In fact, none of these surfaces have a pseudospherical completion; i.e., if one could extend any of these surfaces to be complete, the resulting surface could not be pseudospherical. This is a manifestation of Hilbert's Theorem, to be discussed later on in this thesis.

3.5.1 Associated Sine-Gordon Solutions

Now that we have characterized the pseudospherical surfaces of rotation, it is natural to ask what their corresponding sG solutions are. In order to compute these functions, we must pass either to an asymptotic Chebyshev net, or isothermal-conjugate coordinates, so our parametrizations obtained thus far must be modified.

Let us start with the e-type surface, which is given by a parametrization $\mathbf{x} : (0, 2\pi) \times (-\infty, 0) \to \mathbb{R}^3$ with coordinate functions

$$\mathbf{x}(u, v) = \left(\gamma e^v \cos u, \gamma e^v \sin u, \int_0^v \sqrt{1 - \gamma^2 e^{2t}} dt \right).$$

To obtain an isothermal-conjugate parametrization, consider the change of coordinates given by

$$w = \ln \left(\frac{1 + \sqrt{1 - \gamma^2 e^{2v}}}{\gamma e^v} \right).$$

We see that

$$e^w = \frac{1 + \sqrt{1 - \gamma^2 e^{2v}}}{\gamma e^v} \qquad e^{-w} = \frac{\gamma e^v}{1 + \sqrt{1 - \gamma^2 e^{2v}}},$$

and so performing some algebra gives

$$e^w + e^{-w} = \frac{2}{\gamma e^v},$$

which implies that $\gamma e^v = \operatorname{sech} w$. The coordinate function involving an integral is easily calculated:

$$\int_0^v \sqrt{1 - \gamma^2 e^{2t}} dt = \frac{1}{2} \ln \left| \frac{\sqrt{1 - \gamma^2 e^{2v}} - 1}{\sqrt{1 - \gamma^2 e^{2v}} + 1} \right| + \sqrt{1 - \gamma^2 e^{2v}}.$$

Using an identity from hyperbolic trigonometry then yields $\tanh w = \sqrt{1-\gamma^2 e^{2v}}$. Moreover, a series of computations gives

$$\frac{1}{2}\ln\left|\frac{\sqrt{1-\gamma^2e^{2v}}+1}{\sqrt{1-\gamma^2e^{2v}}-1}\right| - \ln\left|\frac{1+\sqrt{1-\gamma^2e^{2v}}}{\gamma e^v}\right|$$

$$= \frac{1}{2}\left[\ln\left|\frac{\sqrt{1-\gamma^2e^{2v}}+1}{\sqrt{1-\gamma^2e^{2v}}-1}\right| + \ln\left|\frac{\gamma e^v}{\sqrt{1-\gamma^2e^{2v}}+1}\right| - \ln\left|\frac{\sqrt{1-\gamma^2e^{2v}}+1}{\gamma e^v}\right|\right]$$

$$= \frac{1}{2}\left[\ln\left|\frac{\gamma e^v}{\sqrt{1-\gamma^2e^{2v}}-1}\right| - \ln\left|\frac{\sqrt{1+\gamma^2e^{2v}}+1}{\gamma e^v}\right|\right] = \frac{1}{2}\ln\left|\frac{\gamma^2e^{2v}}{\gamma^2e^{2v}}\right| = 0.$$

This gives us another expression for the variable w, namely

$$w = \frac{1}{2}\ln\left|\frac{\sqrt{1-\gamma^2e^{2v}}+1}{\sqrt{1-\gamma^2e^{2v}}-1}\right|,$$

and so we can express the integral in terms of w as

$$\int_0^v \sqrt{1-\gamma^2e^{2t}}dt = \tanh w - w.$$

Relabeling our coordinates back to u and v for convenience, and reflecting the curve about the z-axis, we arrive at the parametrization $\mathbf{x}(u,v) : (0,2\pi)\times\mathbb{R}^+ \to S$ defined by the coordinate functions

$$\mathbf{x}(u,v) = (\operatorname{sech} v\cos u, \operatorname{sech} v\sin u, v - \tanh v).$$

Note that, by this transformation, we have rid ourselves of the constant γ; so the e-type surface is unique up to rigid motion, and is known as the **pseudosphere**. The tangents to the coordinate curves are given by

$$\mathbf{x}_u = (-\operatorname{sech} v\sin u, \operatorname{sech} v\cos u, 0)$$

$$\mathbf{x}_v = (-\operatorname{sech} v\tanh v\cos u, -\operatorname{sech} v\tanh v\sin u, \tanh^2 v),$$

so the first fundamental form is

$$I = \operatorname{sech}^2 v\; du^2 + \tanh^2 v\; dv^2$$

We can also directlty compute the unit normal vector field to the surface:

$$\mathbf{N} = (\tanh v\cos u, \tanh v\sin u, \operatorname{sech} v),$$

and from here the second fundamental form is found to be

$$II = -\operatorname{sech} v\,\tanh v(du^2 - dv^2).$$

Thus, we have moved into isothermal-conjugate parameters. The standard transformation to asymptotic coordinates then gives the first fundamental form in an asymptotic Chebyshev net:

$$I = dx^2 + 2(1 - 2\operatorname{sech}^2(x-t))dxdt + dt^2.$$

So, the c-sG solution corresponding to the pseudosphere is

$$\phi(x,t) = \arccos(1 - 2\operatorname{sech}^2(x-t)),$$

and the analogous l-sG solution is

$$\psi(u,v) = \frac{1}{2}\arccos(1 - 2\operatorname{sech}^2 v).$$

To investigate our other surfaces, we will look for a change of parameters which produces an asymptotic Chebyshev net. This will require some more work; let us first return to the general situation, where we have a chart $\mathbf{x}(u,v) : (0,2\pi) \times U \subset \mathbb{R}^2 \to S$ of a pseudospherical surface of revolution S with coordinate functions

$$\mathbf{x}(u,v) = (f(v)\cos u, f(v)\sin u, g(v)).$$

Assuming that the generating curve is parametrized by arc length, the fundamental forms are given by

$$I = f^2 du^2 + dv^2$$

$$II = -f g_v du^2 + (f_{vv} g_v - f_v g_{vv}) dv^2.$$

Direct computation gives that the Gauss and mean curvature of S yield

$$K = \frac{eg}{EG} \qquad H = \frac{eG + gE}{2EG}.$$

Recall also that the values of principal curvature, k_1 and k_2 at some point on the surface satisfy

$$K = k_1 k_2 \qquad H = \frac{k_1 + k_2}{2}.$$

These equations imply that the principal curvatures on the surface are given by the differentiable functions $k_1 = e/E$ and $k_2 = g/G$. In terms of the functions f and g, we have

$$k_1 = -\frac{g_v}{f} \qquad k_2 = f_{vv} g_v - f_v g_{vv}.$$

For any $p \in S$, the principal directions of S at p are along the orthogonal vectors $\mathbf{x}_u$ and $\mathbf{x}_v$, so any unit vector $v \in T_p S$ has a unique decomposition

$$v = \cos\theta \frac{\mathbf{x}_u}{\sqrt{E}} + \sin\theta \frac{\mathbf{x}_v}{\sqrt{G}},$$

where θ is the oriented angle from $\mathbf{x}_u$ to v. For notational convenience, we will write the normalized vector fields $\mathbf{x}_u$ and $\mathbf{x}_v$ as $\hat{\mathbf{x}}_u$ and $\hat{\mathbf{x}}_v$ Using the fact that $\hat{\mathbf{x}}_u$ and $\hat{\mathbf{x}}_v$ are eigenvectors of the differential of the Gauss map at p with eigenvalues k_1 and k_2, respectively, we can directly compute the normal curvature along v, with

$$\begin{aligned} k_n(v) &= -\langle dN_p(v), v\rangle \\ &= -\langle dN_p(\cos\theta \hat{\mathbf{x}}_u + \sin\theta \hat{\mathbf{x}}_v), \cos\theta \hat{\mathbf{x}}_u + \sin\theta \hat{\mathbf{x}}_v\rangle \end{aligned}$$

$$= -\cos^2\theta\langle dN_p(\hat{\mathbf{x}}_u), \hat{\mathbf{x}}_u\rangle - \sin^2\theta\langle dN_p(\hat{\mathbf{x}}_v), \hat{\mathbf{x}}_v\rangle - 2\sin\theta\cos\theta\langle dN_p(\hat{\mathbf{x}}_u), \hat{\mathbf{x}}_v\rangle,$$

where we have also used self-adjointness in the last line. Notice that the last term vanishes; $dN_p(\hat{\mathbf{x}}_u)$ is along $\hat{\mathbf{x}}_u$, which is orthogonal to $\hat{\mathbf{x}}_v$. Thus, we end up with the expression

$$k_n(v) = k_1\cos^2\theta + k_2\sin^2\theta,$$

which is known as **Euler's Formula**, despite there being many formulas which take exactly this name. To find the asymptotic directions at p, we set the normal curvature to zero, which gives

$$\tan^2\theta = -\frac{k_1}{k_2}.$$

We note that the form of this equation makes plain that the principal directions bisect the asymptotic directions at all hyperbolic points of a surface. Plugging in the values of principal curvature for our pseudospherical surface of revolution, we see that

$$\tan^2\theta = \frac{g_v}{f(f_{vv}g_v - f_v g_{vv})}.$$

By our condition on the Gauss curvature, we know that

$$-1 = K = -\frac{g_v(f_{vv}g_v - f_v g_{vv})}{f},$$

which allows us to simplify our expression, with

$$\tan^2\theta = \frac{g_v^2}{f^2}.$$

Let us now look at the sinh-type surfaces, which can be parametrized by a chart $\mathbf{x}(u, v) : (0, 2\pi)\times(0, b) \to \mathbb{R}^3$ having coordinate functions

$$\mathbf{x}(u, v) = \left(\gamma\sinh v\cos u, \gamma\sinh v\sin u, \int_0^v \sqrt{1 - \gamma^2\cosh^2 t}\; dt\right),$$

where b has been defined earlier, and $|\gamma| < 1$. Our equation yielding the asymptotic directions on this surfaces is then

$$\tan^2\theta = \frac{1 - \gamma^2\cosh^2 v}{\gamma^2\sinh^2 v}.$$

Starting with the positive root, we see that

$$\tan\theta = \frac{\sqrt{1 - \gamma^2\cosh^2 v}}{\gamma\sinh v},$$

from which we obtain

$$\cos\theta = \frac{\gamma\sinh v}{\sqrt{1 - \gamma^2}} \qquad \sin\theta = \sqrt{\frac{1 - \gamma^2\cosh^2 v}{1 - \gamma^2}}.$$

Labeling the asymptotic unit vector field associated to the function $\theta(u, v)$ as $X(u, v)$, we have

$$X(u, v) = \frac{1}{\sqrt{1-\gamma^2}} \left(\mathbf{x}_u + \mathbf{x}_v \sqrt{1 - \gamma^2 \cosh^2 v} \right).$$

Choosing the negative root of the equation, we have that

$$\tan\phi = -\frac{\sqrt{1 - \gamma^2 \cosh^2 v}}{\gamma \sinh v},$$

from which we obtain

$$\cos\phi = \frac{\gamma \sinh v}{\sqrt{1-\gamma^2}} \qquad \sin\phi = -\sqrt{\frac{1 - \gamma^2 \cosh^2 v}{1-\gamma^2}}.$$

Labeling the asymptotic unit vector field associated to the angle function $\phi(u, v)$ as $Y(u, v)$, we have

$$Y(u, v) = \frac{1}{\sqrt{1-\gamma^2}} \left(\mathbf{x}_u - \mathbf{x}_v \sqrt{1 - \gamma^2 \cosh^2 v} \right).$$

Now, we wish to find parameters ξ and η such that, when the chart $\mathbf{x}$ is expressed in these parameters, we have $\mathbf{x}_\xi = X$ and $\mathbf{x}_\eta = Y$; achieving this would mean that $\mathbf{x}(\xi, \eta)$ is an asymptotic Chebyshev net. So, suppose that we have $\mathbf{x}(u(\xi, \eta), v(\xi, \eta))$, where ξ and η are our desired asymptotic Chebyshev parameters. Then, we have the equations

$$X = \mathbf{x}_\xi = \mathbf{x}_u \frac{\partial u}{\partial \xi} + \mathbf{x}_v \frac{\partial v}{\partial \xi}.$$

$$Y = \mathbf{x}_\eta = \mathbf{x}_u \frac{\partial u}{\partial \eta} + \mathbf{x}_v \frac{\partial v}{\partial \eta}.$$

Comparing coefficients, we see that

$$\frac{\partial u}{\partial \xi} = \frac{1}{\sqrt{1-\gamma^2}} \qquad \frac{\partial v}{\partial \xi} = \sqrt{\frac{1 - \gamma^2 \cosh^2 v}{1-\gamma^2}}$$

$$\frac{\partial u}{\partial \eta} = \frac{1}{\sqrt{1-\gamma^2}} \qquad \frac{\partial v}{\partial \eta} = -\sqrt{\frac{1 - \gamma^2 \cosh^2 v}{1-\gamma^2}}.$$

From these equations, we conclude that

$$u = \frac{\xi + \eta}{\sqrt{1-\gamma^2}} + c$$

and $v(\xi, \eta) = v(\xi - \eta)$. The trouble now is expressing the quantity $\langle \mathbf{x}_\xi, \mathbf{x}_\eta \rangle$ in terms of the parameters ξ and η. Well, we have

$$\langle \mathbf{x}_\xi, \mathbf{x}_\eta \rangle = \frac{1}{1-\gamma^2} \left\langle \mathbf{x}_u + \mathbf{x}_v \sqrt{1 - \gamma^2 \cosh^2 v},\ \mathbf{x}_u - \mathbf{x}_v \sqrt{1 - \gamma^2 \cosh^2 v} \right\rangle$$

$$= \frac{\gamma^2 \sinh^2 v - 1 + \gamma^2 \cosh^2 v}{1-\gamma^2}$$

$$= \frac{\gamma^2 \cosh(2v) - 1}{1 - \gamma^2}.$$

In principle, this can be expressed in terms of ξ and η, but we see from the equations above that such an expression necessarily contains an elliptic integral inside the argument of the cosh function, since the expression describing the parameter v in terms of ξ and η involves an elliptic integral. Thus, the c-sG solution corresponding to the sinh-type surface, obtained by applying arccos to the function above expressed in parameters ξ and η, is not expressible in terms of elementary functions.

The story is the same for the cosh-type surfaces; the associated sG solution will involve an elliptic integral term.

Let us return to our sG solution associated to the e-type surface, or pseudosphere. We found that the c-sG solution was

$$\phi(x, t) = \arccos(1 - 2\text{sech}^2(x - t)),$$

where the domain is restricted so that the function is everywhere differentiable, in particular, we require that $x - t > 0$. The form of this solution should be somewhat striking: it is a wave! Indeed, for fixed t, the curve tends quickly to 0 for large x, and changing the parameter t causes this curve to move at uniform velocity. Although not defined on $\mathbb{R}$, this function can be extended to a global, smooth solution to c-sG, which also exhibits wave character, although the function approaches 2π as $x \to -\infty$ rather than 0. This extension is a single soliton solution to sG, and we will find later that there are n soliton solutions for any $n \in \mathbb{N}$, which consist of n waveforms called 'kinks' which go mostly unharmed upon interaction with each other, just as in the case of soliton solutions to the KdV equation discussed in chapter 1. In the next chapter, we will produce machinery which, in principle, allows one to find all soliton solutions to sG through mostly algebraic means. A corollary to this machinery is that all soliton solutions to sG are expressible in terms of elementary functions, so that the cosh- and sinh-type surfaces are not soliton surfaces. This may be surprising considering their simple geometry, but the fact that the domain of the generating curves for these surfaces are bounded in $\mathbb{R}$ makes clear that they should not correspond to sG solutions with smooth extensions to $\mathbb{R}$, and the solitons of sG must, by definition, be globally defined. Although we did not produce the sG solutions for the sinh- and cosh-type surfaces, we can say something about their character with the following result, which does not seem to appear anywhere in the literature.

Proposition 3.5.1. *A pseudospherical surface S is contained in a pseudospherical surface of revolution if and only if there is an l-sG solution representing S which depends non-trivially on exactly one variable.*

Proof. Suppose we have a pseudospherical surface of revolution S. A theorem of Eisenhart tells us that surfaces of revolution all admit isothermal-conjugate parametrizations [5], and these are also parametrizations in lines of curvature. We have already seen that the coefficients of the fundamental forms of an isothermal-conjugate parametrization for a pseudospherical surface yield an associated l-sG solution for S by appropriate manipulations. Thus, we have that any l-sG solution associated to S can be expressed as a function of the coefficients of the fundamental forms of S expressed in isothermal-conjugate parameters. Supposing that we an isothermal-conjugate parametrization $\mathbf{x}(u, v) : U \subset \mathbb{R}^2 \to S$, we can assume without loss of generality that v parametrizes the generating curve, and u is the parameter corresponding to rotation about the axis of symmetry, since these are exactly the principal directions of S. Since rotations about the axis of symmetry are isometries, we have that the fundamental form coefficients at $\mathbf{x}(u, v)$ are equal to those at $\mathbf{x}(u + s, v)$, where $s > 0$ is such that $(u + s, v) \in U$. Thus, the fundamental form coefficients in this parametrization do not depend on u, so that any associated l-sG solution so obtained is a function of v only. The same argument works if S is a proper subset of some pseudospherical surface of revolution.

To prove the converse, suppose that the pseudospherical surface S can be constructed by the l-sG solution $\phi(u, v)$ which depends non-trivially on the variable v only. Then, the surface S has an isothermal-conjugate parametrization in which the fundamental forms read

$$I = \cos^2 \phi du^2 + \sin^2 \phi dv^2$$

$$II = \sin \phi \cos \phi(-du^2 + dv^2),$$

and the Christoffel symbols are now given by

$$\Gamma^1_{11} = 0 \qquad \Gamma^1_{12} = -\phi_v \tan \phi \qquad \Gamma^1_{22} = 0$$

$$\Gamma^2_{11} = \phi_v \cot \phi \qquad \Gamma^2_{12} = 0 \qquad \Gamma^2_{22} = \phi_v \cot \phi,$$

since all partials with respect to u give 0 by assumption. First, we will examine the curves $\alpha(v) = \mathbf{x}(u_0, v)$, which are lines of curvature for each u_0 in the parametrization's domain. With our knowledge of the surface S, the Frenet frame is easily obtained. The unit tangent vector to the curve is simply $t = \mathbf{x}_v/\sqrt{G}$, so that

$$t_v = \frac{\mathbf{x}_{vv}}{\sqrt{G}} - \frac{G_v}{2G^{3/2}}\mathbf{x}_v$$

$$= \frac{1}{\sin \phi}(\Gamma^1_{22}\mathbf{x}_u + \Gamma^2_{22}\mathbf{x}_v + gN) - \frac{2\phi_v \sin \phi \cos \phi}{2 \sin^3 \phi}\mathbf{x}_v = (\cos \phi)N.$$

By definition, the curvature of the curve α is given by $\kappa = \cos \phi$, and the unit normal n to the curve is N, the normal to the surface. The binormal for the curve b is then

$$b = t \wedge n = \frac{\mathbf{x}_v}{\sqrt{G}} \wedge N = \frac{\mathbf{x}_u}{\sqrt{E}}.$$

The torsion of the curve α is given by the equation $b_v = \tau n$, computing gives

$$b_v = \frac{\mathbf{x}_{uv}}{\sqrt{E}} - \frac{E_v}{2E^{3/2}} = \frac{1}{\cos \phi}(\Gamma^1_{12}\mathbf{x}_u + \Gamma^2_{12}\mathbf{x}_v) + \frac{\phi_v \sin \phi \cos \phi}{\cos^3 \phi}\mathbf{x}_u = 0.$$

So the curve α satisfies $\tau \equiv 0$, which is to say that α is a plane curve. Notice that the curvature and torsion of α did not depend on u_0, so that all $u = const.$ curves on S have the same curvature and torsion.

Thus, all such curves are identical, differing only by rigid motion in $\mathbb{R}^3$. As for the curve $\beta(u) = \mathbf{x}(u, v_0)$, the unit tangent vector to the curve is $\mathbf{x}_u/\sqrt{E}$, so that

$$t_u = \frac{\mathbf{x}_{uu}}{\sqrt{E}} - \frac{E_u}{2E^{3/2}}\mathbf{x}_u.$$

Since the fundamental forms are functions only of v, the second term is 0, and we have that

$$t_u = \frac{1}{\cos\phi}(\Gamma^1_{11}\mathbf{x}_u + \Gamma^2_{11}\mathbf{x}_v + eN) = \phi_v\frac{\mathbf{x}_v}{\sqrt{G}} - (\sin\phi)N.$$

Thus, the curvature of β is $\kappa = \sqrt{\phi_v^2 + \sin^2\phi}$, which is constant since it only depends on v. To compute the torsion of β, we first see that its binormal is given by

$$b = \frac{\mathbf{x}_u}{\sqrt{E}} \wedge \left(\frac{\phi_v}{\sqrt{\phi_v^2 + \sin^2\phi}}\frac{\mathbf{x}_v}{\sqrt{G}} - \frac{\sin\phi}{\sqrt{\phi_v^2 + \sin^2\phi}}N \right) = \frac{\mathbf{x}_v + \phi_v N}{\sqrt{\phi_v^2 + \sin^2\phi}}.$$

Taking a derivative yields

$$b_u = \frac{1}{\sqrt{\phi_v^2 + \sin^2\phi}}(\mathbf{x}_{uv} + \phi_v N_u) = \frac{1}{\sqrt{\phi_v^2 + \sin^2\phi}}\left(\Gamma^1_{12}\mathbf{x}_u + \Gamma^2_{12}\mathbf{x}_v + \phi_v \sin\phi \frac{\mathbf{x}_u}{\sqrt{E}} \right).$$

Plugging in values gives $b_u = 0$, which implies that $\tau \equiv 0$. The $v = const.$ curves then have constant curvature and no torsion, which implies that they are pieces of circles.

What we have is that S can be generated by rotating a plane curve α about some piece of a circle β, and we see that we may extend β to be an entire circle, and we may translate and rotate the curve α about this circle in such a way that we extend the domain of the parameter u without altering the fundamental forms. With this modification, rotating S about the line perpendicular to the plane spanned by β and intersecting its center corresponds to a transformation which only affects the u parameter. Since this does not affect the fundamental form coefficients and is a diffeomorphism, it is an isometry. We have therefore found an axis of symmetry for the extension of S, so that S must be contained in a pseudospherical surface or revolution. □

The result is not particularly deep since it only concerns surfaces of a very small collection which we have classified, but it exemplifies the manner in which the geometry of a pseudospherical surface is related to the form of its corresponding sG solutions.

3.6 Curves of Constant Torsion

We now see a slightly different approach to obtaining the sG by studying the binormal motion of curves of constant torsion. A motivation for this approach is the following classical result from surface theory:

Theorem. *(Beltrami-Enneper) The torsion τ of an asymptotic curve on a surface whose curvature is nowhere zero is given by*

$$|\tau| = \sqrt{-K}.$$

Note that no complex numbers arise in equation above: a surface has asymptotic directions at some point p if and only if $K \leq 0$. This means that the asymptotic curves on a pseudospherical surface all have constant torsion. In light of this result, we could view a parametrization of a pseudospherical surface in asymptotic coordinates as describing the motion of a curve of constant torsion, so that these surfaces are really just the figure swept out by the motion of such a curve. The only restriction on the motion of the curve is that it be along its binormal. To see that one always obtains a pseudospherical surface from such a process, suppose that we have a curve of constant torsion undergoing binormal motion through 3-space. This can be described by a differentiable map $\mathbf{x}(s,t) : U \subset \mathbb{R}^2 \to \mathbb{R}^3$, where the restriction $\mathbf{x}(s,t_0)$ describes a curve of constant torsion, and the parameter t describes the rigid motion of this curve. At each fixed t_0, the Frenet frame of the curve is given by

$$\mathbf{t}_s = \kappa\mathbf{n}$$

$$\mathbf{n}_s = \tau\mathbf{b} - \kappa\mathbf{t}$$

$$\mathbf{b}_s = -\tau\mathbf{n},$$

and we wish for this frame to remain orthonormal as t varies. Differentiating the identities

$$\langle \mathbf{t}, \mathbf{n} \rangle = 0$$

$$\langle \mathbf{n}, \mathbf{b} \rangle = 0$$

$$\langle \mathbf{t}, \mathbf{b} \rangle = 0$$

with respect to the parameter t gives us that

$$\langle \mathbf{t}_t, \mathbf{n} \rangle = -\langle \mathbf{t}, \mathbf{n}_t \rangle = 0$$

$$\langle \mathbf{t}_t, \mathbf{b} \rangle = -\langle \mathbf{t}, \mathbf{b}_t \rangle = \alpha,$$

$$\langle \mathbf{n}_t, \mathbf{b} \rangle = -\langle \mathbf{n}, \mathbf{b}_t \rangle = \beta,$$

where $\alpha, \beta : U \to \mathbb{R}$ are differentiable. Note that the inner product $\langle \mathbf{t}_t, \mathbf{n} \rangle$ is zero because at any given point on the curve, the vector $\mathbf{t}_t$ represents the direction in which the curve is travelling, which is along the binormal of the curve, so is perpendicular to the normal vector $\mathbf{n}$. Thus, we can describe the t derivatives of our orthonormal frame as

$$\mathbf{t}_t = \alpha\mathbf{b}$$

$$\mathbf{n}_t = \beta\mathbf{b}$$

$$\mathbf{b}_t = -\alpha\mathbf{t} - \beta\mathbf{n}.$$

Since we require that derivatives with respect to each variable commute, we obtain expressions for $\mathbf{t}_{st} = \mathbf{t}_{ts}$, $\mathbf{n}_{st} = \mathbf{n}_{ts}$, and $\mathbf{b}_{st} = \mathbf{b}_{ts}$ from which we can deduce the relations

$$\alpha_s = \kappa\beta$$

$$\beta_s = -\kappa\alpha$$

$$\kappa_t = -\alpha\tau.$$

In particular, the value $\alpha^2 + \beta^2$ is independent of s, since we have

$$(\alpha^2 + \beta^2)_s = 2\alpha(\kappa\beta) + 2\beta(-\kappa\alpha) = 0.$$

Thus, we can define $\alpha^2 + \beta^2 = \gamma^2(t)$, which is truly a function of t only, and thus

$$\alpha = \gamma \sin\phi \qquad \beta = \gamma\cos\phi,$$

where ϕ is a differentiable function of both s and t. Then the relation $\alpha_s = \kappa\beta$ becomes $\phi_s = \kappa$, and thus

$$\kappa_t = \phi_{st} = -\tau\gamma\sin\phi.$$

We have assumed that the torsion is constant, and in particular we can set $\tau \equiv \pm 1$. In addition, we can simply assume that the curve is traveling at constant unit speed, so that $\gamma \equiv \pm 1$, and changing the sign of γ just changes the direction of motion of the curve. In the case that $\tau = 1$ for example, we choose $\gamma = -1$, and obtain

$$\phi_{st} = \sin\phi.$$

To see how this translates to the situation on surfaces, we put our relations into matrix form, with

$$\begin{pmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{pmatrix}_s = \begin{pmatrix} 0 & \phi_s & 0 \\ -\phi_s & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}\begin{pmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{pmatrix} \qquad \begin{pmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{pmatrix}_t = \begin{pmatrix} 0 & 0 & -\sin\phi \\ 0 & 0 & -\cos\phi \\ \sin\phi & \cos\phi & 0 \end{pmatrix}\begin{pmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{pmatrix}.$$

As before, this system is solvable for smooth functions $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$ comprising the Frenet frame of a curve of constant torsion satisfying binormal motion if and only if ϕ solves c-sG. We also see that, if we send $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}(s,t) \to \{\mathbf{i}, \mathbf{j}, \mathbf{k}\}(x,t)$, and $\phi(s,t) \to -\phi(x,t)$, we obtain the same 3×3 linear representation of section 3.4 derived for an arbitrary pseudospherical surface. Thus, our linear system above is equivalent to the Christoffel system for a pseudospherical surface, having the same compatibility condition.

3.7 Hilbert's Theorem

This section contains ideas from differential and Riemannian geometry which have not been covered in the introductory section on curves and surfaces, as well as more technical points of general topology. For a discussion of the basic theory, see Chapter 5 of [4].

Now that we have established the sG correspondence with pseudospherical surfaces, we should wonder how results about one correspond to the other. In our case, we will use the geometry of pseudospherical surfaces to understand the structure of sG associated to its status as a soliton equation, but it feels equally natural to attempt to understand pseudospherical surfaces by translating geometric theorems to statements about the character of solutions to sG. Perhaps the richest testing ground for this exploration is the following important theorem of Hilbert:

Theorem 3.7.1. *A complete Riemannian* 2*-manifold with constant negative curvature cannot be isometrically immersed in* $\mathbb{R}^3$.

Here, an immersion is a differentiable map between manifolds whose differential is everywhere injective. We do not require that the map itself be injective; in particular, an isometric immersion of a Riemannian 2-manifold $S \to \mathbb{R}^3$ could have some self-intersections. Although there are multiple notions of curvature in Riemannian geometry, they all coincide in dimension 2 to what is essentially the Gaussian curvature (which we saw does not require any reference to the ambient space). The only distinction to be made is that the Riemannian metric on an arbitrary 2-manifold need not be induced by some differentiable immersion of the manifold into $\mathbb{R}^3$ with the standard Euclidean metric. Indeed, this statement displays a class of Riemannian 2-manifolds which lack this property, most notably the Lobachevsky plane and other models of hyperbolic geometry in dimension 2.

To discuss Hilbert's Theorem, some initial remarks are in order. First, we note that if $\phi : M \to N$ is an immersion of differentiable manifolds, and N is equipped with a Riemannian metric, then we can induce a metric on M which makes ϕ an isometric immersion. To see this, take $p \in M$ and $U \subset M$ a neighborhood of p such that $\phi|_U : U \to \phi(U)$ is a diffeomorphism. Since the differential of ϕ is injective at p, we are justified in defining the metric at on M at p as

$$\langle v, w \rangle_p := \langle d\phi_p(v), d\phi_p(w) \rangle_{\phi(p)}$$

where $v, w \in T_pM$. It is routine to check that this actually defines a Riemannian metric on M, from which it immediately follows that ϕ is an isometric immersion.

Now, we take M a complete, Riemannian 2-manifold with constant negative curvature, which we may assume to be -1, and assume the existence of an isometric immersion $\psi : M \to \mathbb{R}^3$. For any $p \in M$, the exponential map $\exp_p : T_pM \to M$ is a local diffeomorphism, in particular an immersion, so we can induce a metric on T_pM such that this is an isometric immersion. We then have an isometric immersion $\psi \circ \exp_p : T_pM \to \mathbb{R}^3$ by composition. The problem is then reduced to showing that there is no isometric immersion of $\mathbb{R}^2 \to \mathbb{R}^3$ given that $\mathbb{R}^2$ is equipped with a metric for which the curvature is constantly equal to -1.

What does this say about the sG? Thus far, we have constructed, from the hypothesis that there exists an isometric immersion of a complete Riemannian 2-manifold with constant curvature -1 into $\mathbb{R}^3$, an isometric immersion $\phi : S \to \mathbb{R}^3$ of a Riemannian 2-manifold S such that S is diffeomorphic to $\mathbb{R}^2$ and has constant curvature -1. Since S is not presented as a regular surface, we do not a priori have any notion of asymptotic directions on S. However, we can define some notion of asymptotic directions on S as follows: for each $p \in S$, choose a neighborhood $U \subset S$ of p such that $\phi|_U : U \to \phi(U)$ is a diffeomorphism. Then $\phi(U)$ is a pseudospherical surface, so that there are two distinct asymptotic directions for $\phi(U)$ at $\phi(p)$. Since the differential of ϕ is an isomorphism here, we can take unit vectors $u', v' \in T_{\phi(p)}\phi(U)$ along the asymptotic directions, and pull them back to vectors $u, v \in T_pS$ via the inverse of the differential of ϕ at $\phi(p)$. The directions defined by the span of the vectors u and v in T_pS then define what we will call the asymptotic directions of S at p. We may define asymptotic curves and the second fundamental form of S similarly: the local extrinsic geometry of the isometrically immersed 'surface' induces an extrinsic geometry on S.

Proposition 3.7.1. *For each $p \in S$, there exists an asymptotic Chebysev net $\mathbf{x} : U \subset \mathbb{R}^2 \to S$ about p, where asymptotic is used in the sense defined above.*

Proof. Choose $V \subset S$ a neighborhood of p such that $\phi|_V : V \to \phi(V)$ is a diffeomorphism, so is an isometry by construction. Since $\phi(p)$ is a point of the pseudospherical surface $\phi(V)$, we have already seen that there exists an asymptotic Chebyshev net $\mathbf{y}(u, v) : U \subset \mathbb{R}^2 \to \phi(V)$ about $\phi(p)$. We then have that $\phi^{-1} \circ \mathbf{y} : \mathbf{y}^{-1}(\phi(V)) \to S$ is an asymptotic Chebyshev net of S about p. □

We now see that the isometric immersion induces local sG solutions on S. The remaining thrust is to show that completeness of S implies the existence of a single asymptotic Chebyshev net covering S.

Proposition 3.7.2. *An arc-length parametrized asymptotic curve $\alpha : (-\epsilon, \epsilon) \to S$ can be extended to an arc-length parametrized asymptotic curve $\tilde{\alpha} : \mathbb{R} \to S$.*

Proof. Suppose not, so that there is some maximal connected domain $(a, b) \subset \mathbb{R}$ such that $\alpha : (a, b) \to S$ is defined. Note that the domain must be open; if it contained an endpoint a or b, then we could extend the domain according to the existence of local asymptotic Chebyshev nets demonstrated in the previous proposition. We then have that $\alpha(t)$ is defined for all $a < t < b$, but not for $t = b$. Taking some sequence $\{t_n\}_{n\in\mathbb{Z}^+}$ of reals converging to b such that $a < t_n < b$ for all $n \in \mathbb{Z}^+$, we see that the sequence $\{\alpha(t_n)\}_{n\in\mathbb{Z}^+}$ is a Cauchy sequence in S since our curve is parametrized by arc length. By completeness, this sequence converges to some $q \in S$. Thus, we may define $\alpha(b) = q$, contradicting the maximality of the domain (a, b). □

Now, we define a map $\mathbf{x} : \mathbb{R}^2 \to S$ as follows: choosing a basepoint $p_0 \in S$, there are two asymptotic directions at p_0, from which we can define the traces of the asymptotic curves α and β intersecting p_0, which are diffeomorphic to $\mathbb{R}$ by the above proposition. We then choose two unit vectors e_1 and e_2 at p_0 in some fixed orientation tangent to the traces of α and β, respectively. For any $(u, v) \in \mathbb{R}$, we first travel along the curve α a distance u beginning at p_0, the direction of travel being determined by the vector e_1, to reach the point $q \in S$. There are two asymptotic directions of S at q, one of which is along α, the other of which corresponds to the trace of some asymptotic curve γ. Note the vectors e_1 and e_2 in the tangent space $T_{p_0}S$ extend to unique asymptotic vector fields along α which are differentiable, the extension of e_1 being everywhere tangent to α. Thus, from the differentiable extension of e_2 along α, we obtain a vector $e_2' \in T_qS$ which lies along the curve γ. We then travel along γ a distance v beginning at q, the direction of travel being determined by the vector e_2', to arrive at the point which we define to be $\mathbf{x}(u, v)$.

It is not difficult to see that, if we fix $v_0 \in \mathbb{R}$, the map $\mathbf{x}(u, v_0) : \mathbb{R} \to S$ is an asymptotic curve parametrized by arc length, and the same holds for the restriction which fixes some $u_0 \in \mathbb{R}$.

Proposition 3.7.3. *The map $\mathbf{x} : \mathbb{R}^2 \to S$ is a surjective local diffeomorphism.*

Proof. To show that $\mathbf{x}$ is a local diffeomorphism, just notice that, for every point $q \in S$, there exists an asymptotic Chebyshev net parametrizing a neighborhood U of q. By construction, the restriction of $\mathbf{x}$ to some $V \subset \mathbb{R}^2$ such that $\mathbf{x}(V) \subset U$ agrees with this parametrization on $\mathbf{x}(V)$ up to some differentiable homeomorphism of the domain, so the restriction $\mathbf{x}|_V : V \to S$ is a diffeomorphism onto its image.

To prove surjectivity, we first note that $\mathbf{x}(\mathbb{R}^2)$ is open in S. This is because we can choose a collection of open sets $\{U_\alpha\}_{\alpha \in J}$ covering $\mathbb{R}^2$ such that $\mathbf{x}|_{U_\alpha} : U_\alpha \to S$ is a diffeomorphism onto its image for all $\alpha \in J$. Thus, $\mathbf{x}(U_\alpha)$ is open in S for all $\alpha \in J$, and $\mathbf{x}(\mathbb{R}^2) = \bigcup_{\alpha \in J} \mathbf{x}(U_\alpha)$, which is open in S.

Assume that $\mathbf{x}$ is not surjective. Then $\mathbf{x}(\mathbb{R}^2)$ is an open, proper subset of S, so there is some point q lying on the boundary of $\mathbf{x}(\mathbb{R}^2)$, which is necessarily not contained in $\mathbf{x}(\mathbb{R}^2)$. Now, consider a neighborhood U of q which is the image of an asymptotic Chebyshev net on S. Since q is a limit point of $\mathbf{x}(\mathbb{R}^2)$, there is some $p \in \mathbf{x}(\mathbb{R}^2) \cap U$. Since p and q both lie in the image of an asymptotic Chebyshve net, the point q can be reached from p by traveling a finite distance along at most two asymptotic curves of S. However, since $p \in \mathbf{x}(\mathbb{R}^2)$, this implies that $q \in \mathbf{x}(\mathbb{R}^2)$: we can travel from p to q by moving a finite distance along at most two asymptotic curves, so q must lie in the image of $\mathbf{x}$ by construction of the map. Thus, $\mathbf{x}$ must be surjective. □

We have one final statement to prove to arrive at the sG version of Hilbert's Theorem

Proposition 3.7.4. *The map $\mathbf{x}$ is a diffeomorphism.*

This will take some work, and we will begin with a statement from general topology.

Lemma 3.7.1. *If $f : \tilde{X} \to X$ is a closed, surjective, local homeomorphism, then f has the path lifting property.*

Throughout the proof, we will adopt the convention $I = [0, 1]$. Recall that a closed map is one which sends closed sets to closed sets, and a map $f : \tilde{X} \to X$ has the path lifting property if for every continuous map $\alpha : I \to X$, there is a lift, i.e. a map $\tilde{\alpha} : I \to \tilde{X}$ such that $f \circ \tilde{\alpha} = \alpha$.

Proof. Let $\alpha : I \to X$ be a continuous map with $\alpha(0) = x_0$. Choosing $\tilde{x}_0 \in f^{-1}(x_0)$, there is a neighborhood U of $\tilde{x}_0$ such that $f|_U : U \to f(U)$ is a homeomorphism. Then we see that the preimage

$\alpha^{-1}(f(U) \cap \alpha(I))$ is open in I and contains 0, so there is some $\epsilon \in (0, 1]$ such that $\alpha([0, \epsilon)) \subset f(U)$, and we can apply the local inverse of f here to obtain a partial lift $\tilde{\alpha} : [0, \epsilon) \to \tilde{X}$ such that $\tilde{\alpha}(0) = \tilde{x}_0$.

Now, assume that the domain of our lift $\tilde{\alpha}$ beginning at $\tilde{x}_0$ cannot be extended to all of I. We will take up the case in which the maximal domain of $\tilde{\alpha}$ is open in I. We have that $\tilde{\alpha} : [0, r) \to \tilde{X}$ cannot be defined at r, where $r \in (0, 1)$. Letting $p = \alpha(r)$, choose $\{U_\beta\}_{\beta \in J}$ an open cover of $f^{-1}(p)$, such that $f|_{U_\beta} : U_\beta \to f(U_\beta)$ is a homeomorphism for all $\beta \in J$. I claim that the U_β's can be chosen so that they are pairwise disjoint. If this were not the case, then there would be some $\tilde{p} \in f^{-1}(p)$ such that, for all neighborhoods V of $\tilde{p}$, we have that $V \cap (f^{-1}(p) \setminus \{\tilde{p}\})$ is non-empty, but this means that there is no neighborhood V of $\tilde{p}$ on which $f|_V : V \to X$ is injective, contrary to assumption. Thus, we assume that the collection $\{U_\beta\}_{\beta \in J}$ is pairwise disjoint, and we have that $W = \sqcup_\beta U_\beta$ is open in $\tilde{X}$ and contains $f^{-1}(p)$. If, for some $\delta < r$, we had that $f^{-1}(\alpha(\delta)) \subset W$, then we would have that one of our local lifts of α to the sets $\{U_\beta\}_{\beta \in J}$ would intersect our original lift $\tilde{\alpha}$ on some interval containing $\tilde{\alpha}(\delta)$, and since this lift contains a point of $f^{-1}(p)$, we could extend the lift $\tilde{\alpha}$ to a domain containing r. Assuming that this is not the case, we then have that $f(\tilde{X} \setminus V)$ is closed in X, since we assumed the map to be closed, and for all $s < \delta$, there is some point of $f^{-1}(\alpha(s))$ contained in $\tilde{X} \setminus V$. From such points and by continuity of f, we can construct a sequence in $f(\tilde{X} \setminus V)$ which converges to p, so that $p \in f(\tilde{X} \setminus V)$ by closedness, but this contradicts the fact that $f^{-1}(p) \subset V$.

The case in which $\tilde{\alpha} : [0, r] \to X$ cannot be extended for some $r \in (0, 1)$ is easier to handle. In any case, we obtain a contradiction, so that f has the path lifting property. □

We are now ready to prove that $\mathbf{x}$ is a diffeomorphism:

Proof. The map $\mathbf{x} : \mathbb{R}^2 \to S$ is clearly closed by construction, so has the path-lifting property by the above lemma. It remains only to show that $\mathbf{x}$ is injective. Taking $x, y \in \mathbf{x}^{-1}(p)$, there is a path $\tilde{\alpha} : I \to \mathbb{R}^2$ with $\alpha(0) = x$ and $\alpha(1) = y$, for example the path traversing the line segment between x and y. We then obtain $\alpha = \mathbf{x} \circ \tilde{\alpha}$ a loop based at p. Since S is diffeomorphic to $\mathbb{R}^2$, it is simply connected, so α is homotopic to the constant loop $c_p : I \to S$ at p. Given the path-lifting property, it is straightforward to show that $\mathbf{x}$ also lifts path homotopies, so that the nullhomotopy of α lifts to a path-homotopy from $\tilde{\alpha}$ to a constant map, and this is only possible if $x = y$. □

Thus, we have a diffeomorphism $\mathbf{x} : \mathbb{R}^2 \to S$ whose differential is everywhere injective since there are two distinct asymptotic directions at every point of S. We have then parametrized our manifold with one chart, in particular one asymptotic Chebyshev net. From this conclusion, we have the following:

Theorem 3.7.2. *The non-existence of a global c-sG solution avoiding integer multiples of π implies Hilbert's Theorem.*

Proof. Assuming an isometric immersion $\psi : M \to \mathbb{R}^3$ of a complete Riemannian 2-manifold with constant curvature -1, we have found that, for any $p \in M$, the manifold T_pM can be covered by one asymptotic Chebyshev net, where the metric and asymptotic directions are induced by the isometric immersion as detailed above. Thus, the metric for T_pM is

$$I = dx^2 + 2\cos\phi\, dxdt + dt^2,$$

where $\phi : \mathbb{R}^2 \to \mathbb{R}$ is a c-sG solution avoiding integer multiples of π. Assuming that no such solution exists, we obtain a contradiction, proving Hilbert's Theorem. □

Note that the hypothesis is equivalent to the non-existence of a global l-sG solution avoiding integer multiples of $\pi/2$.

Proving such a statement about the non-existence of such sG solutions will not be treated in this thesis. However, a possible method of proof could come from the fact that sG can be derived as the equation of motion for the continuous limit of a system of pendula coupled by springs subject to a uniform gravitational field. In this interpretation, the global c-sG solution avoiding integer multiples of π would imply the possibility of a situation in which these pendula never return to their resting position at any time, nor ever swing completely over their pivot points, which certainly contradicts our intuition of classical mechanics. It is quite remarkable that such a system is at all related to the incompatibility of hyperbolic and Euclidean geometry exemplified by Hilbert's Theorem.

4 Pseudospherical Transformations

So far, we have discussed the basic geometrical content of sG; namely the local correspondence between particular parametrizations of pseudospherical surfaces and local solutions to the equation. Since we are dealing with a nonlinear PDE, we cannot simply take linear combinations of such sG solutions to obtain new ones, and a priori, there is no reason to expect that there is any manner in which one can produce new sG solutions from old. However, the pseudospherical transformations discussed in this chapter allow us to do exactly that.

In the late 19th and early 20th century, many prominent geometers focused their attention on transformations of surfaces, and developed a rich theory of such transformations. For a comprehensive and classical review of the subject, see [7]. Of particular use to us are pseudospherical transformations: methods for transforming one surface of constant negative curvature to another having the same constant negative curvature. Such transformations will allow us to transform known sG solutions to obtain new ones. Something particularly remarkable about the transformations which we will describe is that, given an N-soliton solution to sG (whatever this is), we can produce an $(N+1)$-soliton solution. In this way, we will see that the sG satisfies a *nonlinear superposition principle*, which is a common feature to all soliton equations.

4.1 Line Congruences

Before discussing pseudospherical transformations, we will discuss a geometric notion central to many transformations of surfaces in the classical theory.

Definition 4.1.1. *A **congruence of lines** $L : U \times \mathbb{R} \to \mathbb{R}^3$, where $U \subset \mathbb{R}^2$, is a function expressible as*

$$L(u, v, t) = \alpha(u, v) + tX(u, v),$$

where $\alpha, X : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ are differentiable.

Note that this is also referred to in the literature as a line congruence, but the slight difference in nomenclature will be convenient for the discussions which follow. If X does not vanish anywhere, we can consider X to be a map from $U \subset \mathbb{R}^2$ to S^2; the magnitude of $X(u, v)$ does not change the congruence of lines since t may be any real value. One can understand such a congruence of lines in the following way: the map $\alpha(u, v)$ describes some (not necessarily regular) surface $S \subset \mathbb{R}^3$, and the map $X(u, v)$ is a (not necessarily tangent) unit vector field on S. Thus, the congruence of lines describes a family of lines which emanate from points on S in the direction of the vector field X. If one replaces the parameter t with a smooth function $t(u, v) : U \to \mathbb{R}$, we obtain a new surface S' parametrized by the map

$$\beta(u, v) = \alpha(u, v) + t(u, v)X(u, v).$$

We call S' a **focal surface** of the congruence of lines if the line $L(u, v, t)$ is tangent to S' at $\beta(u, v)$ for all $(u, v) \in U$, and S' is a regular surface. Note that if X is a tangent vector field to the regular surface S parametrized by $\alpha(u, v)$, then S is a focal surface of the congruence of lines L. The tangent plane of S' is spanned by the vector fields $\beta_u = \alpha_u + t_u X + tX_u$ and $\beta_v = \alpha_v + t_v X + tX_v$, so saying that $X(u, v)$ lies in this plane is to say that $\langle \beta_u \wedge \beta_v, X \rangle = 0$, or that the matrix whose column vectors are β_u, β_v, and X has zero determinant. Since β_u and β_v contain the terms $t_u X$ and $t_v X$, respectively, we can discard them from the matrix without altering the determinant by elementary matrix operations. Thus, we really have that the matrix whose column vectors are $\alpha_u + tX_u$, $\alpha_v + tX_v$, and X has zero determinant. This is a quadratic equation in t, so we can find at most two smooth functions $t_1(u, v)$ and $t_2(u, v)$ solving the equation. In the case that two such functions exist, we have two focal surfaces S_1 and S_2 of the congruence of lines L, with parametrizations of the form

$$\beta_1(u, v) = \alpha(u, v) + t_1(u, v)X(u, v)$$

$$\beta_2(u, v) = \alpha(u, v) + t_2(u, v)X(u, v).$$

The two focal surfaces are then diffeomorphic via a map which sends a point $\beta_1(u, v) = p \in S_1$ to a corresponding point $\beta_2(u, v) = p' \in S_2$ by moving from p along the line $\alpha(u, v) + sX(u, v)$ a distance $t_2(u, v) - t_1(u, v)$. This diffeomorphism clearly has the property that the line joining p to p' is tangent to both S_1 and S_2 at p and p', respectively. Such a diffeomorphism is referred to as a **line congruence**. Note that congruences of lines with two focal surface are in one-to-one correspondence with line congruences: we have seen that a congruence of lines defines a diffeomorphism between focal surfaces such that the line joining a point to its image is tangent to both surfaces at these points. Moreover, if we have two surface S_1 and S_2, and a line congruence $f : S_1 \to S_2$, we can define a congruence of lines for which S_1 and S_2 are focal surfaces. To see this, let $\mathbf{x}(u, v)$ be a parametrization for the surface S_1. Then, the diffeomorphism f has the form

$$f(\alpha(u, v)) = \alpha(u, v) + t(u, v)X(u, v),$$

where X is a unit tangent vector field on S_1, and t describes the distance one must travel to reach the point $f(p)$. Thus, we have that the congruence of lines

$$L(u, v, t) = \alpha(u, v) + tX(u, v),$$

has S_1 and S_2 as focal surfaces. We now specialize to a particularly nice line congruence:

Definition 4.1.2. *A line congruence $f : S \to S'$ is called a **pseudospherical line congruence** with constant θ if the angle between the tangent planes T_pS and $T_{f(p)}S'$, is constantly equal to θ, and the length of the line joining p to $f(p)$ is constantly equal to* $\sin\theta$.

Calling such a line congruence pseudospherical is quite suggestive, and comes from the work of Bianchi and Bäcklund to be discussed next.

4.2 The Bianchi Transformation

We now turn to the simplest transformation on pseudospherical surfaces discovered by the prolific Italian geometer Luigi Bianchi.

Theorem 4.2.1. *Let S be a pseudospherical surface, and $f : S \to S'$ a pseudospherical line congruence with constant $\pi/2$ such that the line joining p to $f(p)$ is not along a principal direction for any $p \in S$. Then S' is pseudospherical.*

Although we are considering surfaces with $K \equiv -1$, it is also true in general that, given a surface with $K \equiv -1/\rho$ where ρ is positive, the Bianchi Transformation will yield another surface with constant curvature $-1/\rho$.

Proof. Let $\mathbf{x} : U \to S$ be an isothermal conjugate parametrization of S, so that the coefficients of the fundamental forms may take the values

$$E = \cos^2\phi \qquad F = 0 \qquad G = \sin^2\phi$$

$$e = -\sin\phi\cos\phi \qquad f = 0 \qquad g = \sin\phi\cos\phi,$$

At each point $\mathbf{x}(u, v)$, we have a natural orthonormal basis $\{\mathbf{x}_u/\sqrt{E}, \mathbf{x}_v/\sqrt{G}, N\}$ for $\mathbb{R}^3$ which depends smoothly on the parameters u and v. To obtain a parametrization for our transformed surface S', we let $\theta(u, v)$ be the angle which the line joining corresponding points of the surfaces makes with the $+\mathbf{x}_u/\sqrt{E}$ axis. We then have a corresponding parametrization $\mathbf{y} : U \to S'$ defined by

$$\begin{aligned} \mathbf{y}(u, v) &= \mathbf{x}(u, v) + \cos\theta \frac{\mathbf{x}_u}{\sqrt{E}} + \sin\theta \frac{\mathbf{x}_v}{\sqrt{G}}, \\ &= \mathbf{x}(u, v) + \frac{\cos\theta}{\cos\phi}\mathbf{x}_u + \frac{\sin\theta}{\sin\phi}\mathbf{x}_v, \end{aligned}$$

where we have satisfied the condition that the line joining $\mathbf{x}(u, v)$ to $\mathbf{y}(u, v)$ has length constant equal to $\sin\pi/2 = 1$. The other constraints which the construction impose will manifest as conditions on the function θ. To enforce such conditions, we will first compute the tangents to the parameter curves and the normal vector field on S'. The tangent to the parameter curve $v = \text{const.}$ is given by

$$\begin{aligned} \mathbf{y}_u = \mathbf{x}_u &+ \frac{\phi_u \cos\theta \sin\phi - \theta_u \sin\theta \cos\phi}{\cos^2\phi}\mathbf{x}_u + \frac{\cos\theta}{\cos\phi}(\Gamma^1_{11}\mathbf{x}_u + \Gamma^2_{11}\mathbf{x}_v + eN) \\ &+ \frac{\theta_u \cos\theta \sin\phi - \phi_u \sin\theta \cos\phi}{\sin^2\phi}\mathbf{x}_v + \frac{\sin\theta}{\sin\phi}(\Gamma^1_{12}\mathbf{x}_u + \Gamma^2_{12}\mathbf{x}_v + fN). \end{aligned}$$

Plugging in values of the Christoffel symbols gives

$$\begin{aligned} \mathbf{y}_u = &\left[1 + \frac{\phi_u \cos\theta \sin\phi - \theta_u \sin\theta \cos\phi}{\cos^2\phi} - \phi_u \cos\theta \tan\phi \sec\phi - \phi_v \sin\theta \sec\phi\right]\mathbf{x}_u \\ &+ \left[\frac{\theta_u \cos\theta \sin\phi - \phi_u \sin\theta \cos\phi}{\sin^2\phi} + \phi_v \cos\theta \csc\phi + \phi_u \sin\theta \cot\phi \csc\phi\right]\mathbf{x}_v \\ &- [\cos\theta \sin\phi]\, N. \end{aligned}$$

The terms containing ϕ_u cancel in the first two components, and the terms containing ϕ_v and θ_u share a common factor, so we can write

$$\mathbf{y}_u = \left[1 - \sin\theta \sec\phi(\theta_u + \phi_v)\right]\mathbf{x}_u + \left[\cos\theta \csc\phi(\theta_u + \phi_v)\right]\mathbf{x}_v - [\cos\theta \sin\phi]\, N.$$

Rewriting in the orthonormal basis gives

$$\mathbf{y}_u = [\cos\phi - \sin\theta(\theta_u + \phi_v)]\frac{\mathbf{x}_u}{\sqrt{E}} + [\cos\theta(\theta_u + \phi_v)]\frac{\mathbf{x}_v}{\sqrt{G}} - [\cos\theta \sin\phi]N.$$

A similar computation for the tangent to the coordinate curve $u = \text{const.}$ yields

$$\mathbf{y}_v = [-\sin\theta(\theta_v + \phi_u)]\frac{\mathbf{x}_u}{\sqrt{E}} + [\sin\phi + \cos\theta(\theta_v + \phi_u)]\frac{\mathbf{x}_v}{\sqrt{G}} + [\sin\theta \cos\phi]N.$$

Now, the vector representing an infinitesimal displacement from the points $\mathbf{y}(u, v)$ to $\mathbf{y}(u+du, v+dv)$ is

$$[\cos\phi du - \sin\theta(\theta_u du + \phi_v du + \theta_v dv + \phi_u dv)]\frac{\mathbf{x}_u}{\sqrt{E}}$$

$$+[\sin\phi dv + \cos\theta(\theta_u du + \phi_v du + \theta_v dv + \phi_u dv)]\frac{\mathbf{x}_v}{\sqrt{G}}$$

$$+[\sin\theta\cos\phi dv - \cos\theta\sin\phi du]N.$$

Since our pseudospherical line congruence has constant $\pi/2$, we require that the tangent plane of S' is perpendicular to that of S at corresponding points. Thus, the N component of the normal to S' must vanish everywhere. Moreover, the line joining corresponding points of S and S', defined by

$$\cos\theta\frac{\mathbf{x}_u}{\sqrt{E}} + \sin\theta\frac{\mathbf{x}_v}{\sqrt{G}},$$

must lie in the tangent plane to S'. Letting $N'(u,v)$ be the normal vector to S', these conditions imply that

$$N'(u,v) = \sin\theta\frac{\mathbf{x}_u}{\sqrt{E}} - \cos\theta\frac{\mathbf{x}_v}{\sqrt{G}},$$

where we have fixed an orientation arbitrarily. Our displacement vector must be orthogonal to the normal vector N', so we take the dot product of these vectors and equate it to zero:

$$\sin\theta\cos\phi du - \cos\theta\sin\phi dv - (\sin^2\theta + \cos^2\theta)(\phi_v du + \theta_u du + \phi_u dv + \theta_v dv) = 0.$$

Simplifying a bit and multiplying through by -1 leaves us with

$$(\theta_u + \phi_v - \sin\theta\cos\phi)du + (\theta_v + \phi_u + \cos\theta\sin\phi)dv = 0.$$

This relation holds for any infinitesimal displacement, so that both terms inside the parentheses must be identically zero. Now, the function θ satisfies the conditions

$$\theta_u + \phi_v = \sin\theta\cos\phi$$

$$\theta_v + \phi_u = -\cos\theta\sin\phi.$$

By differentiating the first and second equations with respect to u and v, respectively, and subtracting, we see that

$$\theta_{uu} - \theta_{vv} = \sin\theta\cos\theta,$$

so θ is an l-sG solution. Substituting the above relations into the expressions for $\mathbf{y}_u$ and $\mathbf{y}_v$ gives

$$\mathbf{y}_u = [\cos\phi - \sin^2\theta\cos\phi]\frac{\mathbf{x}_u}{\sqrt{E}} + [\sin\theta\cos\theta\cos\phi]\frac{\mathbf{x}_v}{\sqrt{G}} - [\cos\theta\sin\phi]N$$

$$\mathbf{y}_v = [\sin\theta\cos\theta\sin\phi]\frac{\mathbf{x}_u}{\sqrt{E}} + [\sin\phi - \cos^2\theta\sin\phi]\frac{\mathbf{x}_v}{\sqrt{G}} + [\sin\theta\cos\phi]N,$$

which simplify to

$$\mathbf{y}_u = [\cos^2\theta\cos\phi]\frac{\mathbf{x}_u}{\sqrt{E}} + [\sin\theta\cos\theta\cos\phi]\frac{\mathbf{x}_v}{\sqrt{G}} - [\cos\theta\sin\phi]N$$

$$\mathbf{y}_v = [\sin\theta\cos\theta\sin\phi]\frac{\mathbf{x}_u}{\sqrt{E}} + [\sin^2\theta\sin\phi]\frac{\mathbf{x}_v}{\sqrt{G}} + [\sin\theta\cos\phi]N.$$

Now we can compute the coefficients of the first fundamental form for S', which we will denote E', F', and G'. Using the fact that our basis is orthonormal (recall that $F = 0$), we have

$$E' = \cos^4\theta\cos^2\phi + \sin^2\theta\cos^2\theta\cos^2\phi + \cos^2\theta\sin^2\phi$$

$$= \cos^2\theta.$$

$$F' = \sin\theta\cos^3\theta\sin\phi\cos\phi + \sin^3\theta\cos\theta\sin\phi\cos\phi - \sin\theta\cos\theta\sin\phi\cos\phi$$

$$= 0.$$

$$G' = \sin^2\theta\cos^2\theta\sin^2\phi + \sin^4\theta\sin^2\phi + \sin^2\theta\cos^2\phi$$

$$= \sin^2\theta.$$

To emphasize how nicely things have worked out, we note here that, given the original surface S with first fundamental form coefficients

$$E = \cos^2\phi \qquad F = 0 \qquad G = \sin^2\phi,$$

we see that the first fundamental coefficients for S' in the corresponding parametrization are

$$E' = \cos^2\theta \qquad F' = 0 \qquad G' = \sin^2\theta,$$

and by direct computation, we also see that the second fundamental form coefficients for S' are

$$e = -\sin\theta\cos\theta \qquad f = 0 \qquad g = \sin\theta\cos\theta.$$

Thus, parametrizing our surface S in lines of curvature, we see that the corresponding parametrization for S' given by the line congruence is also in lines of curvature. This means that line of curvatures on S are sent to lines of curvature on S' by f, and the same holds true for asymptotic lines (although we do not actually know that S' has asymptotic lines yet since we have not computed its curvature). Such a line congruence which sends asymptotic lines to asymptotic lines is called a W-congruence, named after Julius Weingarten, and plays an important role in the transformations of surfaces with non-positive curvature. Here we see why the requirement that θ is never along a principal direction should be enforced; it means that the image of θ lies in the interval $(0, \pi/2)$, so the transformed surface is everywhere regular. The Christoffel symbols for S' will of course be given by

$$\Gamma^1_{11} = -\theta_u\tan\theta \qquad \Gamma^1_{12} = -\theta_v\tan\theta \qquad \Gamma^1_{22} = -\theta_u\tan\theta$$

$$\Gamma^2_{11} = \theta_v\cot\theta \qquad \Gamma^2_{12} = \theta_u\cot\theta \qquad \Gamma^2_{22} = \theta_v\cot\theta.$$

Plugging these values into the Gauss equation will give us a value for the Gauss curvature on the surface S' :

$$(\theta_u\cot\theta)_u - (\theta_v\cot\theta)_v - \theta_v^2 + \theta_u^2\cot^2\theta - \theta_v^2\cot^2\theta + \theta_u^2 = -\cos^2\theta K.$$

Simplifying the expression gives us

$$(\theta_{uu} - \theta_{vv}) \cot \theta = -\cos^2 \theta K,$$

and since we know that θ is an l-sG solution, the result is $K \equiv -1$, so that S' is pseudospherical. $\square$

The most miraculous geometrical result here is that the meaning of our function θ is two-fold. In addition to its definition as describing the oriented angle which the line joining corresponding points of S and S' makes with the $+\mathbf{x}_u/\sqrt{E}$ axis, the function θ is also an l-sG solution describing an isothermal-conjugate parametrization for S'. On the analytical side of things, we also see that we have discovered a set of coupled differential equations with a remarkable property. In particular, if we have $\phi, \psi : U \to \mathbb{R}^2 \to \mathbb{R}$ smooth functions satisfying

$$\psi_u + \phi_v = \sin \psi \cos \phi$$

$$\psi_v + \phi_u = -\cos \psi \sin \phi,$$

then both ψ and ϕ must be solutions to l-sG. This is seen by computing $\psi_{uu} - \psi_{vv}$ and $\phi_{uu} - \phi_{vv}$ and using the fact that $\partial^2/\partial v \partial u = \partial^2/\partial u \partial v$. Thus, if ϕ is a known function, we can integrate the relations to obtain a new l-sG solution ψ. With an appropriate change of variables, the equations derived become Riccati in ψ, as remarked in [6]. Thus, we must only find one solution, and the general solution is obtainable by integration.

Another important note here is that, although we assumed the existence of a pseudospherical line congruence, we actually *constructed* this line congruence by means of integrating the relations to obtain the function which we called θ.

4.3 The Bäcklund Transformation

Soon after Bianchi's discovery of the transformation described above, the Swedish geometer Albert Victor Bäcklund realized that the construction could be generalized in such a way that one can produce a parametrized family of sG solutions from a known seed solution. In particular, he proved the following:

Theorem 4.3.1. *Let S be a pseudospherical surface, and $f : S \to S'$ a pseudospherical line congruence with constant $\sigma \in (0, \pi)$ such that the line joining p to $f(p)$ is not along a principal direction. Then S' is pseudospherical.*

The only difference between the transformations of Bäcklund and Bianchi is that, while Bianchi assumed that the tangent planes of S and S' were orthogonal at corresponding points, Bäcklund tells us that we may allow the angle between tangent planes to be any constant in $(0, \pi)$, and we still obtain another pseudospherical surface. Thus, while Bianchi only generates one new pseudospherical surface (up to a constant of integration), Bäcklund allows us to find uncountably many.

Proof. Although the definition of a pseudospherical line congruence says that the line joining corresponding points of S and S' must be constantly equal to $\sin\sigma$, we will see that this assumption actually follows from the fact that the angle between tangent planes is constantly equal to σ. Thus, we will just assume that the line joining corresponding points of S and S' is some positive constant λ. Once again, we will let $\mathbf{x}(u,v) : U \to S$ be an isothermal-conjugate parametrization. The corresponding parametrization of the surface S' is then

$$\mathbf{y}(u,v) = \mathbf{x}(u,v) + \lambda\cos\theta\frac{\mathbf{x}_u}{\sqrt{E}} + \lambda\sin\theta\frac{\mathbf{x}_v}{\sqrt{G}},$$

where θ is defined similarly to that in the proof for Bianchi's Transformation. The tangents to the parameter curves are given by

$$\mathbf{y}_u = \left[\cos\phi - \lambda\sin\theta(\theta_u + \phi_v)\right]\frac{\mathbf{x}_u}{\sqrt{E}} + \left[\lambda\cos\theta(\theta_u + \phi_v)\right]\frac{\mathbf{x}_v}{\sqrt{G}} - \left[\lambda\cos\theta\sin\phi\right]N.$$

$$\mathbf{y}_v = \left[-\lambda\sin\theta(\theta_v + \phi_u)\right]\frac{\mathbf{x}_u}{\sqrt{E}} + \left[\sin\phi + \lambda\cos\theta(\theta_v + \phi_u)\right]\frac{\mathbf{x}_v}{\sqrt{G}} + \left[\lambda\sin\theta\cos\phi\right]N.$$

Just as for the Bianchi case, we express the displacement in $\mathbb{R}^3$ which results from an infinitesimal displacement in parameter space. The resulting vector is

$$\begin{aligned}&\left[\cos\phi du - \lambda\sin\theta(\phi_v du + \theta_u du + \phi_u dv + \theta_v dv)\right]\frac{\mathbf{x}_u}{\sqrt{E}}\\ &+ \left[\sin\phi dv + \lambda\cos\theta(\phi_v du + \theta_u du + \phi_u dv + \theta_v dv)\right]\frac{\mathbf{x}_v}{\sqrt{G}}\\ &+ \left[\lambda\sin\theta\cos\phi dv - \lambda\cos\theta\sin\phi du\right]N.\end{aligned}$$

Now, since the unit normal of S and S' meet at constant angle σ, the N component of our normal vector field on S', denoted N', should take the constant value $\cos\sigma$. Imposing the additional condition that the line joining corresponding points of our surfaces lies in the tangent plane of S' gives us the formula

$$N' = \sin\sigma\sin\theta\frac{\mathbf{x}_u}{\sqrt{E}} - \sin\sigma\cos\theta\frac{\mathbf{x}_v}{\sqrt{G}} + \cos\sigma N.$$

We again take the dot product of the normal vector and our infinitesmal displacement and set it to zero. We obtain similar expressions as in the case of Bianchi. The du and dv terms of this equation must vanish everywhere, which leads us to the relations

$$\lambda\sin\sigma(\theta_u + \phi_v) = \sin\sigma\sin\theta\cos\phi - \lambda\cos\sigma\cos\theta\sin\phi$$

$$\lambda\sin\sigma(\theta_v + \phi_u) = -\sin\sigma\cos\theta\sin\phi + \lambda\cos\sigma\sin\theta\cos\phi.$$

We can find the promised expression which relates λ and σ by differentiating these equations with respect to v and u, respectively, and subtracting the results. The left-hand side reads $\lambda\sin\sigma(\phi_{vv} - \phi_{uu})$, which can be expressed as $-\lambda\sin\sigma\sin\phi\cos\phi$ since ϕ solves l-sG. After a bit of algebra, the equation becomes

$$-\lambda\sin\sigma\sin\phi\cos\phi = \frac{1}{\lambda\sin\sigma}(\lambda^2\cos^2\sigma - \sin^2\sigma)\sin\phi\cos\phi.$$

By simplifying we see that $\lambda^2 = \sin^2\sigma$. Since we are assuming that both quantities are non-negative, we obtain $\lambda = \sin\sigma$, recovering what was before required by the definition of pseudospherical line congruence. Our relations now read

$$\sin\sigma(\theta_u + \phi_v) = \sin\theta\cos\phi - \cos\sigma\cos\theta\sin\phi$$

$$\sin\sigma(\theta_v + \phi_u) = -\cos\theta\sin\phi + \cos\sigma\sin\theta\cos\phi.$$

Notice that, as we had hoped, setting $\sigma = \pi/2$ obtains the analogous relations of the Bianchi Transformation. Now, if we differentiate the first and second equations with respect to u and v, respectively, and subtract the results, more of the same algebra as before gives us an anticipated relation, namely

$$\theta_{uu} - \theta_{vv} = \sin\theta\cos\theta.$$

So far, we have found a more general relation which generates new l-sG solutions from old. To confirm that our new surface is pseudospherical, we compute the first fundamental form coefficients. Using the equations obtained above, the tangent vectors to the coordinate curves are

$$\mathbf{y}_u = [\cos\theta(\cos\theta\cos\phi + \cos\sigma\sin\theta\sin\phi)]\frac{\mathbf{x}_u}{\sqrt{E}}$$

$$+[\cos\theta(\sin\theta\cos\phi - \cos\sigma\cos\theta\sin\phi]\frac{\mathbf{x}_v}{\sqrt{G}} - [\sin\sigma\cos\theta\sin\phi]N$$

$$\mathbf{y}_v = [\sin\theta(\cos\theta\sin\phi - \cos\sigma\sin\theta\cos\phi)]\frac{\mathbf{x}_u}{\sqrt{E}}$$

$$+[\sin\theta(\sin\theta\sin\phi + \cos\sigma\cos\theta\cos\phi)]\frac{\mathbf{x}_v}{\sqrt{G}} + [\sin\sigma\sin\theta\cos\phi]N.$$

By taking dot products, we retrieve our coefficients. Thankfully for us, we obtain

$$E' = \cos^2\theta \qquad F' = 0 \qquad G' = \sin^2\theta.$$

Since we have the exact same expressions as in Bianchi, we need not repeat the computations; the Gauss equation shows that $K \equiv -1$. □

As before with Bianchi, we have discovered that if $\phi, \psi : U \subset \mathbb{R}^2 \to \mathbb{R}$ are smooth and satisfy

$$\sin\sigma(\psi_u + \phi_v) = \sin\psi\cos\phi - \cos\sigma\cos\psi\sin\phi$$

$$\sin\sigma(\psi_v + \phi_u) = -\cos\psi\sin\phi + \cos\sigma\sin\psi\cos\phi,$$

then ϕ and ψ solve l-sG. Thus, if ϕ is known, then we obtain a family of l-sG solutions parametrized by $\sigma \in (0, \pi)$. This represents the first non-trivial example of what are now simply called Bäcklund Transformations for differential equations. Although there is not a strict consensus on definition, we will give a definition as follows. Suppose that $\phi(x, t)$ solves some differential equation, say

$$P[\phi(x, t)] = 0,$$

where P is a differential operator which can be nonlinear and involve both x and t derivatives, and suppose that $\psi(x,t)$ satisfies

$$B_i[\phi(x,t), \psi(x,t)] = 0,$$

where the B_i's are a finite number of differential operators involving only first order derivatives. Then, if ψ is found to also satisfy $P[\psi(x,t)] = 0$, then the operators B_i are referred to as a Bäcklund Transformation for the differential equation P.

A familiar example of a Bäcklund Transformation is the Cauchy-Riemann system; for real valued functions $u(x,y)$ and $v(x,y)$, if

$$u_x = v_y \qquad u_y = -v_x,$$

then u and v satisfy the Laplace equation:

$$u_{xx} + u_{yy} = 0 \qquad v_{xx} + v_{yy} = 0,$$

and in particular, the function $f(x+iy) = u(x,y) + iv(x,y)$ is holomorphic. Bäcklund Transformations are most useful in the case that the corresponding differential equation is nonlinear, so this example is rather trivial, but gives a feel for the machinery nonetheless. Note that, in both the Cauchy-Riemann system and the classical Bäcklund Transformation derived in this section, our Bäcklund Transformations have the property that, so long as ϕ and ψ satisfy a system of coupled differential equations, they must both satisfy some other equation, in our cases sG or the Laplace equation. This need not be the case in general; we might require that one of the functions is already known to satisfy the differential equation for the other to as well.

This piece of machinery has proven quite useful in the study of differential equations, and as we have remarked before, all soliton equations enjoy Bäcklund Transformations. We can certainly take pride in the fact that the origin of such transformations rests in the geometry of pseudospherical surfaces.

4.3.1 The Bäcklund Transformation in Asymptotic Coordinates

From hereon, we will refer to both the transformation of pseudospherical surfaces and the system of differential equations relating solutions to sG as a Bäcklund Transformation, with the hopes that context will spare the reader any confusion. Although our isothermal-conjugate parameters were useful in the preceding constructions by providing a smoothly varying orthonormal frame on our surface, we have already seen that something is gained by taking the asymptotic perspective, i.e. the geometric significance of sG solutions as describing the angle between asymptotic curves in an asymptotic Chebyshev net. It should also be noted that the Bäcklund Transformation written in asymptotic coordinates is often considered as canonical for sG.

The aim of this section is then to produce the differential equations derived in the theorem of Bäcklund in their corresponding c-sG form. We will begin by supposing that $\phi/2$ is an l-sG solution.

Then the Bäcklund Transformation for the function $\theta/2$ reads

$$\sin\sigma(\theta_u/2 + \phi_v/2) = \sin\theta/2\cos\phi/2 - \cos\sigma\cos\theta/2\sin\phi/2$$

$$\sin\sigma(\theta_v/2 + \phi_u/2) = -\cos\theta/2\sin\phi/2 + \cos\sigma\sin\theta/2\cos\phi/2,$$

where $\sigma \in (0, \pi)$. Applying the coordinate transformation $x = u+v$ and $t = u-v$, we obtain the relations

$$\frac{\sin\sigma}{4}(\theta_x + \theta_t + \phi_x - \phi_t) = \sin\theta/2\cos\phi/2 - \cos\sigma\cos\theta/2\sin\phi/2$$

$$\frac{\sin\sigma}{4}(\theta_x - \theta_t + \phi_x + \phi_t) = -\cos\theta/2\sin\phi/2 + \cos\sigma\sin\theta/2\cos\phi/2.$$

We can add and subtract these equations, respectively, and arrive at

$$\frac{\sin\sigma}{2}(\theta_x + \phi_x) = (1+\cos\sigma)(\sin\theta/2\cos\phi/2 - \cos\theta/2\sin\phi/2)$$

$$\frac{\sin\sigma}{2}(\theta_t - \phi_t) = (1-\cos\sigma)(\sin\theta/2\cos\phi/2 + \cos\theta/2\sin\phi/2).$$

We can divide through by $\sin\sigma$, and the terms containing σ are then equal to $\cot\sigma/2$ for the first equation and $\tan\sigma/2$ for the second equation. We can use trigonometric identities to simplify the second terms of the right-hand sides of these equations. Setting $\lambda = \tan\sigma/2$, and multiplying through by 2 gives

$$\theta_x + \phi_x = \frac{2}{\lambda}\sin\left(\frac{\theta-\phi}{2}\right)$$

$$\theta_t - \phi_t = 2\lambda\sin\left(\frac{\theta+\phi}{2}\right).$$

Once can easily check that ϕ and θ both solve c-sG if these relations are satisfied. What is new here is that our coupled differential equations are ordinary, and we now have a transformation parameter λ which may take any positive real as a value, the special case of the Bianchi transformation corresponding to $\lambda = 1$.

Although the classical Bäcklund Transformation required us to work with regular surfaces, we see that the corresponding differential equations apply just as well to sG solutions which do not represent surfaces, namely the constant solutions. Thus, we should try to perform the simplest possible Bäcklund Transformation, beginning with the seed solution $\phi \equiv 0$. In asymptotic coordinates, our relations read

$$\theta_x = \frac{2}{\lambda}\sin\frac{\theta}{2}$$

$$\theta_t = 2\lambda\sin\frac{\theta}{2}.$$

Integration of these relations then returns the function

$$\theta(x,t) = 4\arctan\left(e^{x/\lambda + \lambda t}\right),$$

where we throw away the constant of integration since it does not change the resulting surface obtained in a non-trivial manner. These are the 1-solitons of c-sG. Setting $\lambda = 1$, we obtain the "Bianchi Transformation of a line," which can be written as

$$\phi = 4\arctan\left(e^{\upsilon}\right),$$

where we have set $\upsilon = x + t$. To place this into a previously encountered form, we notice that

$$e^{\upsilon} = \tan(\phi/4),$$

from which it follows that

$$\operatorname{sech}\upsilon = \sin(\phi/2),$$

and so we see that

$$\cos\phi = 1 - 2\operatorname{sech}^2\upsilon,$$

where our use of a half-angle identity means that this relation only holds for non-positive reals. Since this corresponds, up to coordinate transformations, to the l-sG solution for the pseudosphere, we see that the Bianchi transformation of a line is the pseudosphere, where the soliton solution obtained is globally defined, and extends the pseudosphere such that the generating curve is defined over the non-zero reals.

4.4 Bianchi's Permutability Theorem

Upon the integration of the relations obtained in the previous section, we obtain a family of sG solutions parametrized by λ. One could just as easily wish to perform subsequent Bäcklund Transformations to obtain other families of solutions. This involves solving systems of ODEs which quickly become complicated. However, a striking result of Bianchi tells us that we can compute the result of iterated Bäcklund transformations by algebra alone, given that we have enough seed solutions. The result also tells us that iterative Bäcklund transformations are commutative. To avoid verbosity, we will write $B_\sigma S$ to represent the surface obtained by performing the Bäcklund Transformation with parameter σ to the surface S.

Theorem 4.4.1. *(Bianchi's Permutability Theorem) Let S be a pseudospherical surface, and $\sigma_1, \sigma_2 \in (0, \pi)$. Then $B_{\sigma_2} B_{\sigma_1} S$ and $B_{\sigma_1} B_{\sigma_2} S$ represent the same pseudospherical surface. Moreover, the sG solution corresponding to $B_{\sigma_2} B_{\sigma_1} S$ can be obtained by algebra alone, given that the sG solutions corresponding to S, $B_{\sigma_1} S$, and $B_{\sigma_2} S$ are known.*

Note that we are assuming the existence of Bäcklund Transformations, which only holds locally in general.

Proof. The proof is simple when considering c-sG solutions. We will assume that ϕ is the corresponding c-sG solution for the surface S, and that θ_1, θ_2 are those corresponding to $B_{\sigma_1} S$ and $B_{\sigma_2} S$ respectively. We will allow ω to be the c-sG solution corresponding to $B_{\sigma_2} B_{\sigma_1} S$ and $B_{\sigma_1} B_{\sigma_2} S$. If we find that such a function exists, then we will see that both surfaces have the same c-sG solution, since Bäcklund Transformations are unique up to constants of integration, which shows that the surfaces are indeed the same. Letting $\lambda_1 = \tan\sigma_1/2$ and $\lambda_2 = \tan\sigma_2/2$, we have the equations

$$\theta_{1,t} = \phi_t + 2\lambda_1 \sin(\frac{\theta_1 + \phi}{2})$$

$$\theta_{2,t} = \phi_t + 2\lambda_2 \sin(\frac{\theta_2 + \phi}{2})$$

$$\omega_t = \theta_{1,t} + 2\lambda_2 \sin(\frac{\omega + \theta_1}{2})$$

$$\omega_t = \theta_{2,t} + 2\lambda_1 \sin(\frac{\omega + \theta_2}{2})$$

Adding the first and third equations and subtracting the second and fourth from this gives

$$0 = \lambda_1 \left[\sin\left(\frac{\theta_1 + \phi}{2}\right) - \sin\left(\frac{\omega + \theta_2}{2}\right)\right] + \lambda_2 \left[\sin\left(\frac{\omega + \theta_1}{2}\right) - \sin\left(\frac{\theta_2 + \phi}{2}\right)\right].$$

Using a familiar trigonometric identity gives

$$0 = \lambda_1 \sin\left(\frac{\omega + \phi + \theta_2 + \theta_1}{4}\right)\cos\left(\frac{(\omega - \phi) + (\theta_2 - \theta_1)}{4}\right) + \lambda_2 \sin\left(\frac{\omega + \phi + \theta_2 - \theta_1}{4}\right)\cos\left(\frac{(\omega - \phi) - (\theta_2 - \theta_1)}{4}\right).$$

Simplification and more trigonometry yields

$$\begin{aligned}
&\frac{\lambda_2}{\lambda_1}\left[\cos\left(\frac{\omega - \phi}{4}\right)\sin\left(\frac{\theta_2 - \theta_1}{4}\right) - \sin\left(\frac{\omega - \phi}{4}\right)\cos\left(\frac{\theta_2 - \theta_1}{4}\right)\right] \\
&= -\sin\left(\frac{\omega - \phi}{4}\right)\cos\left(\frac{\theta_2 - \theta_1}{4}\right) - \cos\left(\frac{\omega - \phi}{4}\right)\sin\left(\frac{\theta_2 - \theta_1}{4}\right),
\end{aligned}$$

which can be rearranged to the expression

$$\left(\frac{\lambda_2}{\lambda_1} + 1\right)\cos\left(\frac{\omega - \phi}{4}\right)\sin\left(\frac{\theta_2 - \theta_1}{4}\right) = \left(\frac{\lambda_2}{\lambda_1} - 1\right)\sin\left(\frac{\omega - \phi}{4}\right)\cos\left(\frac{\theta_2 - \theta_1}{4}\right).$$

Finally, we arrive at a very nice relation between these c-sG solutions:

$$\tan\left(\frac{\omega - \phi}{4}\right) = \frac{\lambda_2 + \lambda_1}{\lambda_2 - \lambda_1}\tan\left(\frac{\theta_2 - \theta_1}{4}\right).$$

In particular, the function ω does exist and has the algebraic expression

$$\omega = \phi + 4\arctan\left[\frac{\lambda_2 + \lambda_1}{\lambda_2 - \lambda_1}\tan\left(\frac{\theta_2 - \theta_1}{4}\right)\right].$$

□

Given the unproven fact that applying the Bäcklund Transformation to an N-soliton solution to sG returns an $(N+1)$-soliton solution, Bianchi's Permutability Theorem gives a method for obtaining N-soliton solutions from single soliton and constant solutions. Indeed, if we have N distinct single soliton solutions to c-sG, say $\theta_1, ..., \theta_N$, we can apply the formula above to obtain the two-soliton solutions $\phi_1, ..., \phi_{N-1}$, where ϕ_i is obtained from the constant zero solution as well as the single soliton solutions θ_i and θ_{i+1}. We can then iterate this process to obtain an N-soliton solution, and we can do so algebraically, which is quite miraculous considering that we are really producing successively richer solutions to a nonlinear PDE. Of course, these functions will become more and more complicated, but one must admit that algebra is a wonderful alternative to integration of coupled differential equations.

As remarked in section 3.5.1, Bianchi's Permutability Theorem tells us that soliton solutions to sG are expressible in terms of elementary functions, so that the sinh- and cosh-type surfaces of revolution are not soliton surfaces.

As an example, let us write down the expression for the 2-solitons of sG. We begin with two distinct 1-soliton solutions, which have the form

$$\theta_1 = 4\arctan\left(e^{x/\lambda_1 + t\lambda_1}\right)$$

$$\theta_2 = 4\arctan\left(e^{x/\lambda_2 + t\lambda_2}\right),$$

where $\lambda_1 \neq \lambda_2 \neq 0$. Using the formula from the proof and trigonometry, we then see that

$$\omega = 4\arctan\left(\frac{\lambda_2 + \lambda_1}{\lambda_2 - \lambda_1}\frac{e^{x/\lambda_2 + t\lambda_2} - e^{x/\lambda_1 + t\lambda_1}}{1 + e^{x/\lambda_1 + t\lambda_1}e^{x/\lambda_2 + t\lambda_2}}\right),$$

which consists of two 'kinks' traveling at constant speed and interacting with each other by simply summing.

5 Soliton Equations

To make more transparent the relationship between soliton equations and pseudospherical surface theory, we give a brief exposition of the inverse scattering method for solving soliton equations, and remark the connection between this method and the discussions of chapters 3 and 4

5.1 Inverse Scattering for the Schrödinger Equation

We now discuss the basic idea of the inverse scattering transform for the time independent Schrödinger equation

$$\psi_{xx}(x,k) + V(x)\psi(x,k) = k^2\psi(x,k),$$

and suggest how the use of this equation allows one to produce solutions to the KdV equation, the first soliton equation to be solved by inverse scattering. See [3] for a more detailed exposition.

5.1.1 The Direct Spectral Problem

We will first summarize the direct spectral problem for the Schrödinger equation. We begin with a smooth function $u : \mathbb{R} \to \mathbb{R}$ such that the quantity

$$\int_{-\infty}^{\infty} \big(1 + |x|\big)|u(x)|dx$$

is finite. We want to determine the character of solutions to the time independent Schrödinger equation subject to the potential $u(x)$, which reads

$$-\psi_{xx}(x, k) + u(x)\psi(x, k) = k^2\psi(x, k).$$

Solutions to this equation are the eigenfunctions of the Schrödinger operator $d^2/dx^2 + u$, which have eigenvalues $k^2 \in \mathbb{R}$. The spectrum of the Schrödinger operator consists of a continuous component corresponding to $k \in \mathbb{R}$, as well as a finite (possibly empty) collection of discrete negative eigenvalues $\{k_1^2, ..., k_n^2\}$. The eigenfunctions corresponding to the eigenvalues k^2 such that $k \in \mathbb{R}$ are the unique solutions satisfying the asymptotic boundary conditions

$$\psi(x, k) \to T(k)e^{-ikx}, \quad x \to -\infty$$

$$\psi(x, k) \to e^{-ikx} + R(k)e^{ikx}, \quad x \to \infty,$$

where $T(k)$ and $R(k)$ are known as the transmission and reflection coefficients, respectively. The idea here is that a wave is shot in 'from infinity,' and after interaction with the potential u, some of this wave is reflected back from whence it came, and some is transmitted through the potential field, asymptotically approaching a wave (recall that the potential $u(x)$ decays quickly for large $|x|$). We will quote the result that the transmission and reflection coefficients for $k \neq 0$ are given by

$$T(k) = 1 + \frac{1}{2ik}\int_{-\infty}^{\infty} e^{ikx}\psi(x, k)u(x)dx$$

$$R(k) = \frac{1}{2ik}\int_{-\infty}^{\infty} e^{-ikx}\psi(x, k)u(x)dx.$$

We note also that, if k is real, we have the identity $|R(k)|^2 + |T(k)|^2 = 1$, so that knowledge of the reflection coefficients easily gives the transmission coefficients, and vice versa. In the physical interpretation, this is a manifestation of conservation of energy. As for the discrete negative eigenvalues, we have that

$$k_j = ip_j,$$

where the p_j are real and positive, and i is the imaginary unit. The eigenfunctions corresponding to these eigenvalues are known to physicists as 'bound states.' Viewed as a quantum mechanical system, the eigenvalue k_j^2 corresponds to the energy of a particle whose wavefunction is given by the eigenfunction associated to k_j^2. Since this energy is negative, the particle cannot escape the potential, so will not scatter off to infinity. Observe that, if the potential $u(x)$ is everywhere negative, there will be no bound states, and indeed, the number of negative eigenvalues corresponds in a precise sense to the non-negativity of the potential u. From this discussion, it is not surprising that the eigenfunctions associated to negative eigenvalues are square integrable. Denoting the eigenfunction associated to k_j^2 as $f_j(x)$, the asymptotic behavior of f_j is given by

$$f_j(x) \to e^{-p_j x}$$

as $x \to \infty$. By square integrability, we have normalization coefficients $\{\rho_1, ..., \rho_n\}$ given by

$$\frac{1}{\rho_j} = \int_{-\infty}^{\infty} f_n^2(x)dx.$$

We define the **spectral transform** of $u(x)$ as the data

$$S[u] = \{R(k) \text{ for } k \in \mathbb{R}, p_1, ..., p_n, \rho_1, ..., \rho_n\}.$$

The determination of this data from a known potential u is the content of the direct spectral problem.

5.1.2 The Inverse Spectral Problem

The **inverse spectral problem** is the task of recovering the potential $u : \mathbb{R} \to \mathbb{R}$ from a given spectral transform. In particular, if we have a function $R : \mathbb{R} \to \mathbb{C}$ which satisfies $|R(k)| \leq 1$, and

$$\lim_{k \to \pm\infty} R(k) = 0,$$

and given arbitrary positive numbers $p_1, ..., p_n, \rho_1, ..., \rho_n$, can we obtain a smooth, rapidly decaying potential $u(x)$ which has this data as its spectral transform? The answer is yes, and we will state how to construct the potential without proof. The crucial step is the use of the Gel'fand-Levitan-Marchenko equation, which reads

$$K(x, y) + M(x + y) + \int_x^{\infty} K(x, z)M(z + y)dz = 0, \;\; y > x.$$

The function M is given in terms of the spectral transform data by

$$M(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} R(k)e^{ikx}dx + \sum_{j=1}^{n} \rho_j e^{-p_j x}.$$

Here we see a justification as to why the inverse scattering transform is sometimes viewed as a nonlinear analogue to the Fourier transform: the function M is the inverse Fourier transform of the reflection coefficient together with a contribution from the discrete spectrum of the sought after Schrödinger operator. Defining the function

$$w(x) = 2 \lim_{\epsilon \to 0} \left[K(x, x + \epsilon)\right],$$

the potential is simply $u = -w_x$. As an example, consider the scattering data

$$S = \{R \equiv 0, p_1 = p, \rho_1 = \rho\},$$

a reflectionless potential with a single discrete eigenvalue. Then

$$M(x) = \rho e^{-px},$$

and solving the now separable Gel'fan-Levitan-Marchenko equation yields the potential

$$u(x) = -\frac{2p^2}{\cosh^2(px - 1/2\ln(\rho/2p))}.$$

5.1.3 The Inverse Scattering Method

Now, we will briefly describe the application of the Schrödinger scattering problem in solving the initial value problem for the KdV equation. Suppose that we have some function $u_0(x) : \mathbb{R} \to \mathbb{R}$ which is smooth and vanishes sufficiently fast for large $|x|$ so that it may play the role of the potential in the direct scattering problem. Then, we can determine the scattering data here as sketched above, obtaining

$$S(0) = \{R(k,0), p_1(0), ..., p_n(0), \rho_1(0), ..., \rho_n(0)\}.$$

Suppose also that we search for a function $u(x,t)$ satisfying the KdV equation

$$u_t + u_{xxx} - 6uu_x = 0$$

subject to the constraint $u(x,0) = u_0(x)$. It turns our that requiring the potential $u(x,t)$ to evolve in time in accordance with the KdV equation requires that the scattering data evolves in time with

$$R(k,t) = R(k,0)e^{8ik^3t}$$

$$p_n(t) = p_n(0)$$

$$\rho_n(t) = \rho_n(0)e^{8p_n^3t}.$$

For a justification of this fact, see chapter 2 of [3]. Note here that the discrete spectrum of $S(t)$ does not change in time, so the scattering data is said to evolve isospectrally. From here, one can find the scattering data for arbitrary time, yielding

$$S(t) = \{R(k,t), p_1(t), ..., p_n(t), \rho_1(t), ..., \rho_n(t)\}.$$

From this data, one performs inverse scattering at arbitrary time to recover the function $u(x,t)$. Although not true for arbitrary nonlinear evolution equations, this method does produce functions which solve the KdV equation and satisfy $u(x,0) = u_0(x)$, the reasons essentially stemming from the fact that the scattering data $S(t)$ takes a particularly simple form.

This procedure is called the **Inverse Scattering Method** (ISM) for the Schrödinger problem, and can be employed for certain nonlinear equations of the form

$$u_t = F(u, u_x, u_{xx}, ...).$$

The idea, as we have seen, is that from some initial function $u_0(x)$, one applies direct scattering to obtain $S(0)$. Then, the evolution equation allows one to obtain the scattering data at arbitrary time, from which one carries out inverse scattering to obtain the function $u(x, t)$.

From our cursory review of the method for solving the KdV equation, we can now place on firmer footing the idea of a 'soliton.' In particular, an N-soliton for some nonlinear PDE solvable by some form of the ISM is a solution for which the scattering data is such that $R(k, t) = 0$ for all t, and there are exactly N discrete eigenvalues $p_1(0), ..., p_n(0)$. From the scattering evolution written above for the KdV equation, it is clear that the discrete spectrum of the Schrödinger operator does not depend on time, and this is also true generically for equations solvable by this method. This is more-or-less by definition: a soliton equation must be solvable by some form of the ISM for which the discrete spectrum of the associated linear problem evolves isospectrally in the 'time' variable. This isospectral evolution of the scattering data explains why solitons possess the remarkable stability first observed by John Scott Russell in 1845. In addition, the stipulation of a reflectionless potential causes the Gel'fan-Levitan-Marchenko equation to be separable, so that soliton solutions can be produced exactly.

5.2 The AKNS Method and Relations to Surface Theory

From this discussion alone, one is justified in asking how this relates at all to surface theory. Although we have seen solutions to sG claimed to be 'solitons,' we clearly need a different linear problem than the time-independent Schrödinger equation to apply inverse scattering for sG. Indeed, the inverse spectral transform for the Schrödinger problem on the real line can only be applied to a specific class of nonlinear PDEs of the form

$$u_t = F(u, u_x, u_{xx}, ...),$$

in which class sG is not included. However, the AKNS method discovered by Ablowitz, Kaup, Newell, and Segur generalizes the application of inverse scattering considerably, and can be summarized as follows: we seek a pair of linear operators $A(x, t, \lambda)$ and $B(x, t, \lambda)$ such that the nonlinear PDE to be solved takes the form

$$A_t - B_x + [A, B] = 0.$$

The necessary conditions for these operators are the following: that A is associated to the scattering problem $v_x = Av$, the spectral parameter λ is such that $\lambda_t = 0$ (isospectral evolution), the operator B is such that, given v a solution to the scattering problem $v_x = Av$, we also have that $v_t - Bv$ is a solution to the scattering problem, and lastly that $A_t - B_x + [A, B]$ is not a differential operator. In this case, the inverse scattering problem goes through similarly as in the case of the Schrödinger problem. An interesting technical difference is that the eigenvalues of the discrete spectrum of A need not lie on the imaginary axis as in the Schrödinger problem, which gives rise to complex-conjugate pairs of discrete eigenvalues whose soliton solutions, called breathers, are periodic in the variable t. This method was a modification of the so-called Zhakarov-Shabat system for solving the non-linear Schrödinger equation:

$$iu_t + u_{xx} + 2|u|^2 u = 0$$

by employing the method of inverse scattering to the associated spectral problem

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}_x = \begin{pmatrix} -i\lambda & u(x,t) \\ -u^*(x,t) & i\lambda \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix},$$

where * denotes complex conjugation. In the AKNS-method, the pair of operators for the nonlinear Schrödinger equation are

$$A = \begin{pmatrix} -i\lambda & u \\ -u^* & i\lambda \end{pmatrix} \qquad B = \begin{pmatrix} -2i\lambda^2 + i|u|^2 & 2\lambda u + iu_x \\ -2\lambda u^* + iu_x^* & 2i\lambda^2 - i|u|^2 \end{pmatrix}.$$

The AKNS method was originally devised to solve sG [1], and the corresponding operators are

$$A = \begin{pmatrix} -i\lambda & -1/2\phi_x \\ 1/2\phi_x & i\lambda \end{pmatrix} \qquad B = \frac{i}{4\lambda} \begin{pmatrix} \cos\phi & \sin\phi \\ \sin\phi & -\cos\phi \end{pmatrix},$$

which is the linear representation derived in section 3.4 up to a gauge transformation. Thus, the AKNS system for the sG is not some happy accident which mathematicians have stumbled upon: these are equations from classical surface theory. Indeed, the equations $v_x = Av$ and $v_t = Bv$ are simply the Christoffel system of section 2.2.4, up to the $\mathfrak{so}(3)$ - $\mathfrak{su}(2)$ isomorphism and an appropriate gauge transformation, and the compatibility condition $A_t - B_x + [A, B] = 0$ is the Gauss equation. For a more complete review of the AKNS method for solving soliton equations, see chapter 2 of [9].

An even more striking fact is that any soliton equation solvable via the AKNS method describes pseudospherical surfaces, and Bäcklund Transformations can be obtained by the classical transformation of pseudospherical surfaces which we have discussed at length. This was first observed by Sasaki in [13], and lead to the notion of non-linear evolution equations of "pseudospherical type," introduced by Chern and Tenenblat in 1986, and discussed in [11].

Although this fact seems to justify the statement that "soliton theory is surface theory," there are more elementary connections between surface theory and soliton equations which we have not been discussed. For example, the nonlinear Schrödinger equation was derived by Hasimoto in 1972 in connection with the motion of a thin isolated vortex filament moving through an incompressible fluid [10]. In particular, the motion of such a vortex filament draws out a so-called Hasimoto surface, for which the Gauss equation becomes the nonlinear Schrödinger equation. Examples of such connections abound in surface theory; a thorough and extensive exposition of such results is given in [12].

6 Conclusion

In this thesis, we investigated the relationship between pseudospherical surfaces and the sine-Gordon equation. After discussing this simple relationship through the study of pseudospherical surfaces of revolution and Hilbert's Theorem, our exploration of the pseudospherical transformations of Bianchi and Bäcklund led up to discover methods for producing a wealth of solutions to the sine-Gordon equation, including 1- and 2-soliton solutions. We then related the geometric structure of the sine-Gordon equation to its status as a soliton equation, and remarked that the connection between soliton theory and pseudospherical surface theory extends to a large class of nonlinear PDEs.

The relationship between soliton equations and differential geometry is a vast subject which we have only scratched the surface of here. The interested reader is referred to [14] as a starting point to explore more recent advancements and open problems in the field.

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