Chapter 1 - Complex Numbers

A complex number $z$ has the form

$$z =$$

where

We can associate a complex number $z = a + bi$ with

Complex numbers are added in the

If $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$ (where $x_1, x_2, y_1, y_2$ are real numbers), then

$z_1 + z_2 :=$

The product is defined as

$$z_1 z_2 =$$

Properties: Let $z_1, z_2, z_3$ be complex numbers. Then

1. $z_1 + z_2 =$
2. $z_1 z_2 =$
3. $z_1 (z_2 + z_3) =$
4. $z_1 + =$
5. $= z_1$
(6) \( z + 1 - z = 0 \),

where

(7) The set of complex numbers is

(8) \( z_1 + z_2 + z_3 = \)

\( z_1 \cdot z_2 \cdot z_3 = \)

Equality: Two complex numbers \( z_1 = x_1 + iy \), \( z_2 = x_2 + iy \), (also are equal if and only if

**Theorem:** If \( z \neq 0 \) is a complex number, then there is a complex number \( z^{-1} \) such that

**Proof:** Let \( z = x + iy \) where

Suppose \( z \neq 0 \). Then

Observe that

\[ z \cdot ( \quad ) = \]

So \( z \cdot z^{-1} = 1 \), and

\[ z^{-1} = \]

**Example:** Let \( z = 3 + 4i \). Find \( z^{-1} \).
Division: If $z_1, z_2$ are complex numbers, then

$$\frac{z_1}{z_2} = \frac{z_1}{z_2}$$

Note: If $z_2 \neq 0$, then $\frac{1}{z} = \frac{\bar{z}}{|z|^2}$.

Real Part & Imaginary Part

Let $z = x + iy$ where $x, y \in \mathbb{R}$.

$x$ is called the 

$y$ is called the 

We write

Example: Let $z = \frac{1}{2} + 3i$.

Exercise: Prove Property (2).

If $z_1, z_2$ are complex numbers, then $z_1 z_2 = z_2 z_1$.

Proof: Let $z_1, z_2$ be complex numbers. Then

$$z_1 = \frac{1}{2}, \quad z_2 = 3i$$
Exercise: Prove Property (3).
If $z_1, z_2, z_3$ are complex numbers then
$$z_1(z_2 + z_3) = z_1z_2 + z_1z_3.$$
Geometric Interpretation of Complex Addition.

Addition of complex numbers corresponds to

\[ z_1 + z_2 \]

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**Definition** Let \( z = x + iy \) where \( x, y \in \mathbb{R} \).

The conjugate of \( z \), denoted by \( \overline{z} \), is defined by

\[ \overline{z} = x - iy \]

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**Note (1)** \( z \overline{z} = \) \[ z \overline{z} = |z|^2 \]

\[ |z|^2 = \]

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(2) The distance from \( z \) to \( 0 \) is

**Definition** Let \( z = x + iy \) where \( x, y \in \mathbb{R} \).

The modulus (or absolute value) of \( z \) is defined by

\[ |z| = \sqrt{x^2 + y^2} \]

Note \[ |z|^2 = \]

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Example: Let \( z = 3 + 4i \). Find \( \bar{z} \) and \( |z| \).

\[ \overline{z_1 + z_2} = \]
\[ \overline{z_1 z_2} = \]
\[ \text{If } z_3 \neq 0, \quad \frac{1}{z_3} = \]
\[ \text{Re}(z) = \quad \text{Im}(z) = \]
\[ |z_1, z_2| = \]
\[ \text{Re}(z) = \]
\[ \text{Im}(z) = \]
\[ |z_1 + z_2| = \]
\[ \text{PROOF: Let } z = x + iy, \quad z_1 = x_1 + iy_1, \quad z_2 = x_2 + iy_2 \text{ where } \]
\[ x, y, x_1, y_1, x_2, y_2 \in \mathbb{R} \]
\[ \frac{z_1 + z_2}{z_1 + z_2} = \]
\[ \frac{z_1 z_2}{z_1 z_2} = \]
(3) Suppose \( z_2 \neq 0 \). Then

\[
\left( \frac{z_1}{z_2} \right) z_2 =
\]

(4)

\[
\begin{align*}
z &= \\
\bar{z} &= \\
\frac{z}{\bar{z}} &= \\
\frac{z}{z} &= \\
\frac{z - \bar{z}}{z + \bar{z}} &=
\end{align*}
\]

(5)

\[
\left| z_1 \bar{z}_2 \right|^2 =
\]

(6)

\[
\Re(z) =
\]

(7)

\[
\Im(z) =
\]
(c) \( |z_1 + z_2|^2 \)

**Triangle Inequality**
The Polar Form of a Complex Number

Let \( z = x + iy \) \((x,y) \in \mathbb{R}\) \& \( z \neq 0\).

Then the polar form of \( z \) is

where

\[
\begin{align*}
x &= \\
y &= \\
t &= \\
\tan \theta &= \\
\end{align*}
\]

**NOTE:** Given \( z \in \mathbb{C} \) (The set of Complex Numbers), \( z \neq 0 \),

\( r \) is

but \( \theta \) is

Each value of \( \theta \) is called

and is denoted by

The principal value of \( \arg z \) denoted by

is the

Example: Find the polar form of \( z = -\sqrt{3} + i \).
Example Let \( z = -1 - i \). Find \( \text{Arg} \, z \).

Proposition
Suppose \( z = r \left( \cos \theta + i \sin \theta \right) \),
\[ w = \rho \left( \cos \varphi + i \sin \varphi \right) \]
where
Then
\[ zw = \]

Proof:
Corollary. If $z_1, z_2 \neq 0$ then

(i) $\arg(z_1, z_2) =$

(ii) $\arg\left(\frac{z_1}{z_2}\right) =$

Exponential Form
**Definition** For \( \theta \in \mathbb{R} \), define
\[
e^{i\theta} := \text{ }
\]

\[|e^{i\theta}| = \text{ }\]

**Proposition** Let \( \theta, \varphi \in \mathbb{R} \). Then
\[
e^{i(\theta + \varphi)} = \text{ }\]

**Proof** Assume \( \theta, \varphi \in \mathbb{R} \). Then
\[
e^{i(\theta + \varphi)} = \text{ }\]

**Proposition** Let \( \theta \in \mathbb{R} \). If \( z = e^{i\theta} \) then
\[
\frac{z}{z} = \text{ }\] and \( \frac{1}{z} = \text{ }\)

**Proof** Let \( \theta \in \mathbb{R} \). Then
\[
e^{i\theta} e^{i(-\theta)} = \text{ }\]
Proposition. Let $r_1, r_2, \theta_1, \theta_2 \in \mathbb{R}$ with $r_1, r_2 > 0$.

Let $z_1 = r_1 e^{i\theta_1}$, $z_2 = r_2 e^{i\theta_2}$.

Then $z_1 = z_2$ if and only if

The equation $z = re^{i\theta}$ is the parametric equation of a circle centered at $(0, 0)$ and radius $r$ (with $\theta$ orientation).

Note $|z| = r$

De Moivre's Theorem. Let $\theta \in \mathbb{R}$, $n$ be a positive integer.

Let $z = r e^{i\theta}$.

Then

$$(\cos \theta + i \sin \theta)^n =$$

Proof. Let $\theta \in \mathbb{R}$. We prove the statement by $\cdots$
Ex Find $(1 + i)^{100}$
Ex Find all \( z \in \mathbb{C} \) not satisfy \( z^3 = 1 \).
Theorem: Let \( n \) be a positive integer. The equation \( z^n = 1 \) has \( n \) complex solutions.

where \( k = \)

\[ z_k = \]

**NOTE:**
1. These solutions are called \( \omega \).
2. Let \( z_1 = e^{2\pi i/n} = \omega \). Then \( k \) solutions are \( z = \)

Theorem: Let $w$ be an nonzero complex number with polar form

$$w = \rho e^{i\theta} = \rho \cos \theta + i \rho \sin \theta.$$

Let $n$ be a positive integer. Then the equation

$$z^n = w$$

has $n$ complex solutions, namely

$$z = \rho^{1/n} \left( \cos \left( \frac{\theta + 2k\pi}{n} \right) + i \sin \left( \frac{\theta + 2k\pi}{n} \right) \right),$$

where $k = 0, 1, \ldots, n-1$.

Ex: Solve $z^4 = -16$. 


Proof of Theorem