Chapter 2 Analytic Functions

Functions of a Complex Variable

We will consider functions $f : S \to \mathbb{C}$

where $S \subset \mathbb{C}$ is the domain of f.

Examples. Find the domain of the following functions

(i) $f(z) = \frac{1}{z}$ (ii) $f(z) = Arg(z)$ (iii) $f(z) = Arg(\frac{1}{z})$.

Note Let $S \subset \mathbb{C}$. Any function $f : S \to \mathbb{C}$

can be write as

$f(z) =$

where $z = x + iy \in S$.

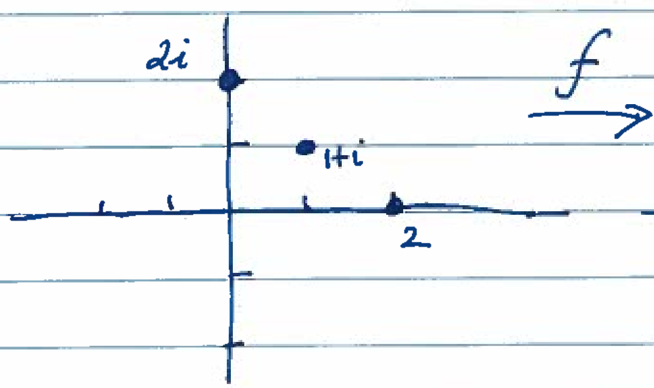
The functions u:

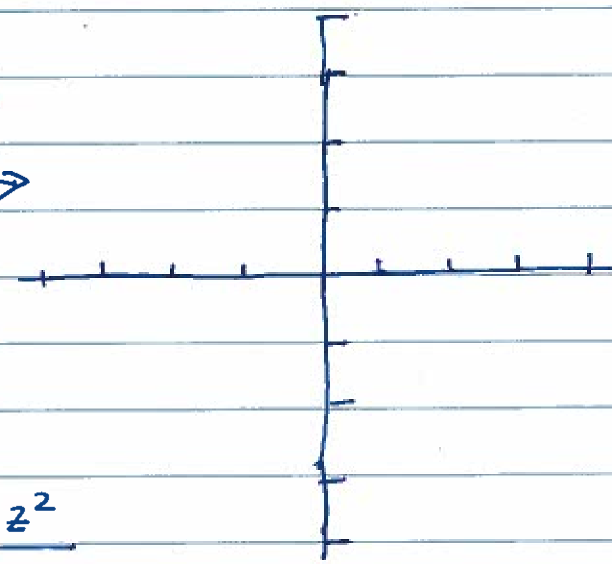
v:

where

Example Write the function $f(z) = \frac{1}{z}$ in the form $f(z) = u(x,y) + i\,v(x,y)$.

The Mapping $w = z^2$





Complete:

z	$f(z) = z^2$
$2i$	
$1+i$	
2	

Ex Let $f(z) = z^2$.

(i) Find the image of the imaginary axis

(ii) Find the image of any vertical line $Re(z) = c$ where c is a non-zero constant. Sketch the image.

(iii) Find the image of the real axis.

(iv) Find the image of any horizontal line $Im(z) = c$ where c is any non-zero constant. Sketch the image.

<u>NOTE:</u> Let $S \subset \mathbb{C}$ & suppose $f: S \to \mathbb{C}$.
Let $A \subset S$. The image of <u>A under the map f</u> is denoted by ___ and is defined by

Let $f: \mathbb{C} \to \mathbb{C}$ by $f(z) = z^2$.

Then $f(z) = (x + iy)^2$ $\quad (z = x+iy)$

$=$

$= u(x,y) + i\,v(x,y)$

where $u(x,y) =$

$v(x,y) =$

(i) Let $z =$ where

Then $f(z) =$

The image of the imaginary axis under f is

(iii) Let $z =$

where

Then $f(z) =$

(iii) Let $z =$

where

Then $f(z) =$

(iv) Let $z =$

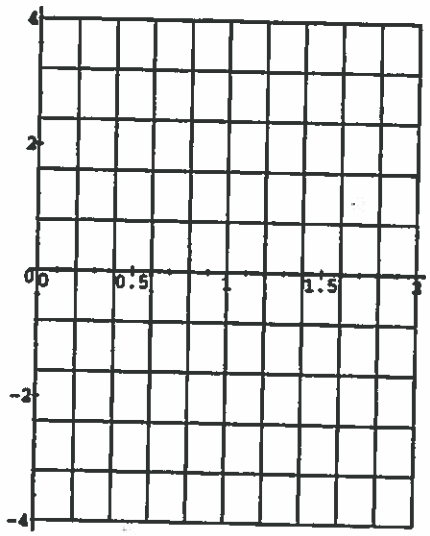
where

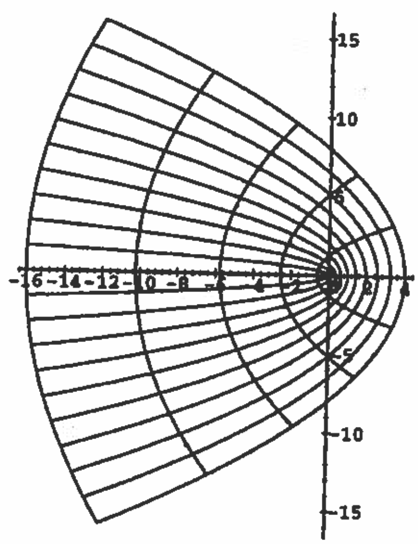
Then $f(z) =$

(p. 6)

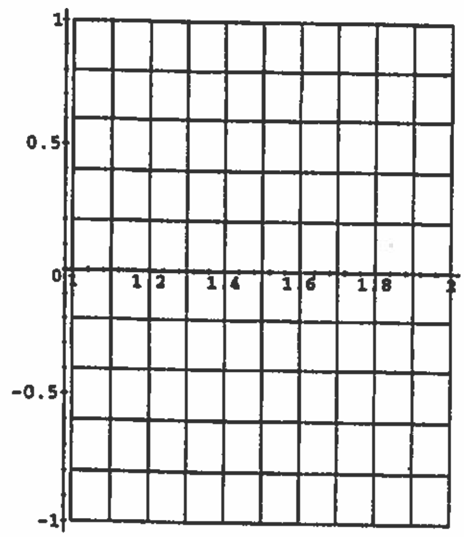
Mappings of Complex Functions

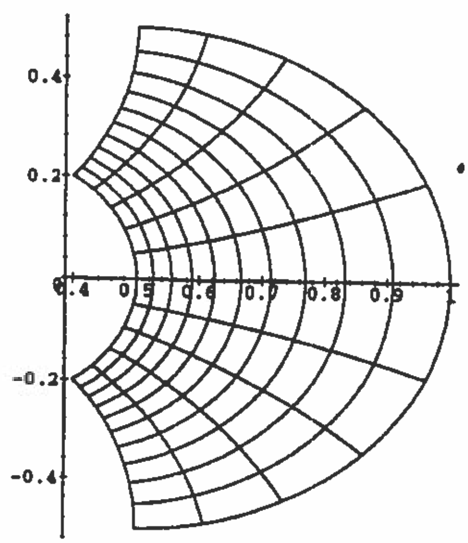
1. $f(z) = z^2$



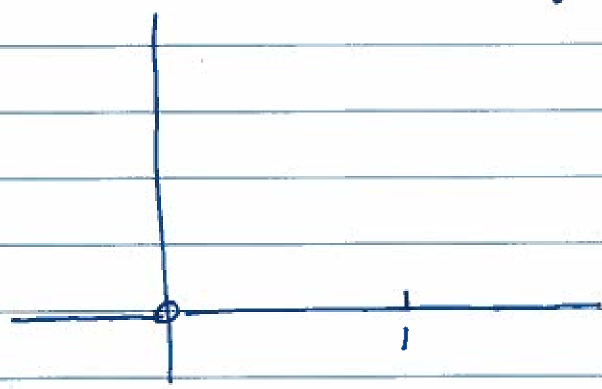


2. $f(z) = \frac{1}{z}$





Example Find the image of the sector
$0 \leq \theta \leq \pi/4$, $0 < r \leq 1$
under the map $f(z) = \frac{1}{z}$.



Let $z \in S$ then
$z =$

where
Then $w = f(z) =$

Example Let $f: \mathbb{C} \to \mathbb{C}$ by $f(z) = e^{\frac{\pi i}{3}} z$.
Describe this map.

Exercise

Under the map $w = \frac{1}{z}$ show that

(i) The image of any horizontal line (except thro 0) is a circle.

(ii) The image of any vertical line (except thro 0) is a circle.

Hint: Try to get u^2+v^2 in terms of u or v.

Limits

(p.10)

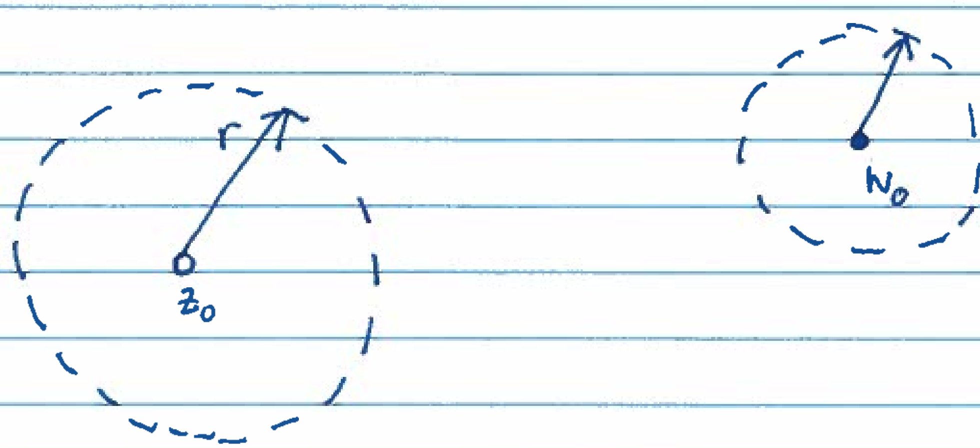
Definition: Let $z_0 \in \mathbb{C}$. A **deleted (punctured) neighborhood** of z_0 has the form

$$D'(z_0, r) = \{ z \in \mathbb{C} : \qquad \}$$

Definition Let $z_0 \in \mathbb{C}$ & suppose $f : D'(z_0, r) \to \mathbb{C}$ for some $r > 0$. Let $w_0 \in \mathbb{C}$. We say the limit of $f(z)$ as z approaches z_0 is w_0 and write

____

if



Example Using the definition show that
$$\lim_{z \to i} (2z+1) = 2i+1.$$

WORKING: Let $\varepsilon > 0$. We want to show that there is a ___
such that

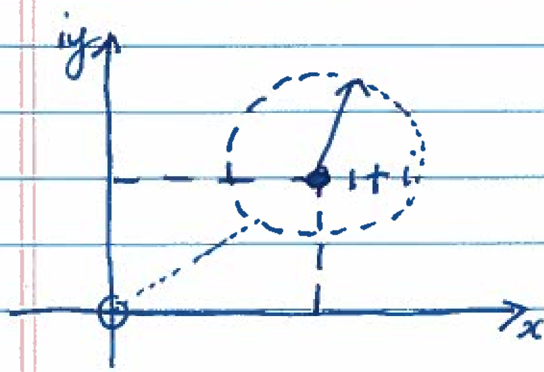
if

PROOF: Let $\varepsilon > 0$ be any fixed positive real number.
Suppose

Example Use the definition to prove that

$$\lim_{z \to 1+i} \frac{1}{z} =$$

WORKING:



PROOF: Let $\varepsilon > 0$ be any fixed positive real number.
Let $\delta =$

Theorems on Limits

Theorem Let f be defined on an deleted neighborhood of $z_0 = x_0 + iy_0$ & let $w_0 = u_0 + iv_0$ where $x_0, y_0, u_0, v_0 \in \mathbb{R}$. ~~Then~~ Let $f(z) = u(x,y) + i\,v(x,y)$ for $z = x+iy$ in domain of f. Then

$$\lim_{z \to z_0} f(z) = w_0$$

if and only if

Example Let $f(z) = \frac{1}{z}$.
Find $\lim_{z \to 1+i} f(z)$.

Proof of Theorem

($\Rightarrow$) Suppose $\lim_{z \to z_0} f(z) = w_0$.

Theorem Suppose $f(z)$ and $g(z)$ are complex-valued functions defined on a deleted neighborhood of z_0, and suppose

$$\lim_{z\to z_0} f(z) = \omega_0 \quad \text{and} \quad \lim_{z\to z_0} g(z) = \omega_1.$$

Then

(1) $\lim_{z\to z_0} \left(f(z)+g(z)\right) =$

(2) $\lim_{z\to z_0} f(z)\,g(z) =$

(3) $\lim_{z\to z_0} \dfrac{f(z)}{g(z)} =$ if

(4) $\lim_{z\to z_0} c\,f(z) =$ if

(5) $\lim_{z\to z_0} z^n =$ if

(6) $\lim_{z\to z_0} P(z) =$

if $P(z) = a_0 + a_1 z + \cdots + a_n z^n$ is a

(7) $\lim_{z\to z_0} R(z) =$

if $R(z) = \dfrac{P(z)}{Q(z)}$ is a and .

THE POINT AT INFINITY

Suppose $z_0 \in \mathbb{C}$ and $f(z)$ is defined on a deleted neighborhood of z_0.

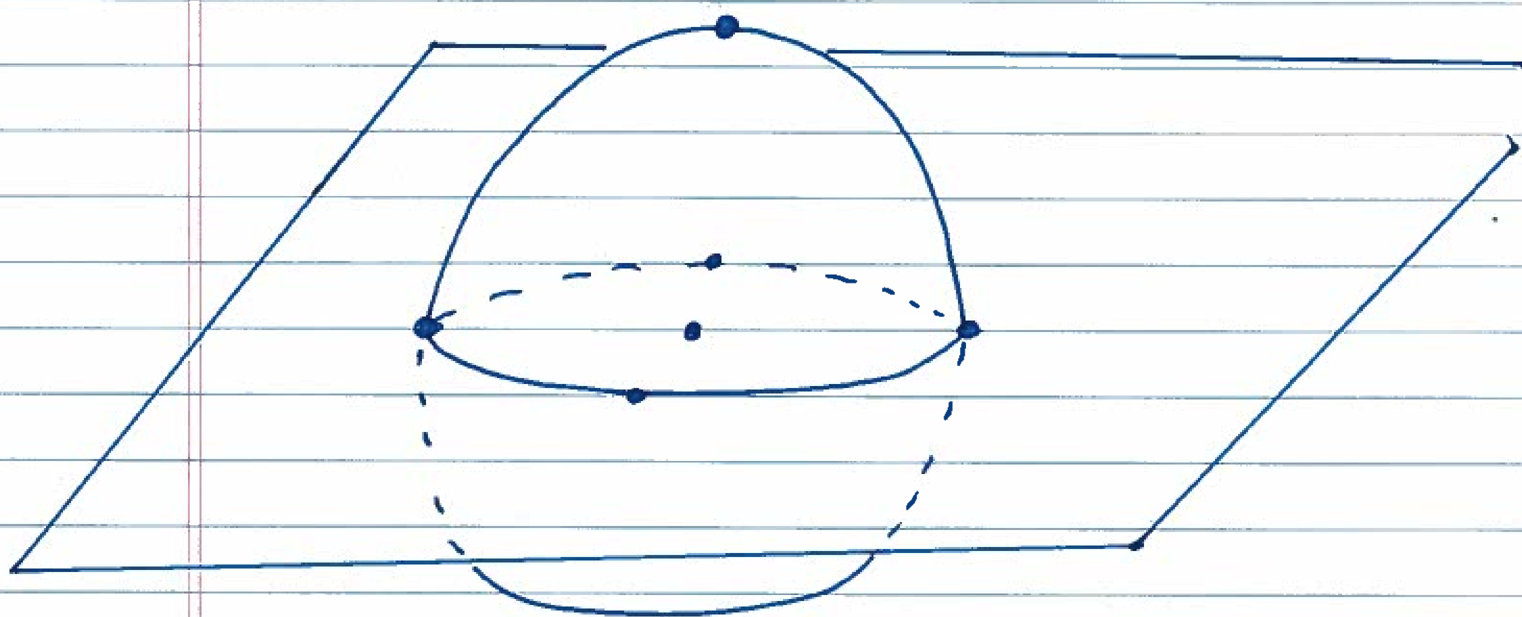
Definition. We say (write) $\lim_{z \to z_0} f(z) = \infty$

if

NOTE: This holds iff

Hence $\lim_{z \to z_0} f(z) = \infty$ iff

The Riemann Sphere



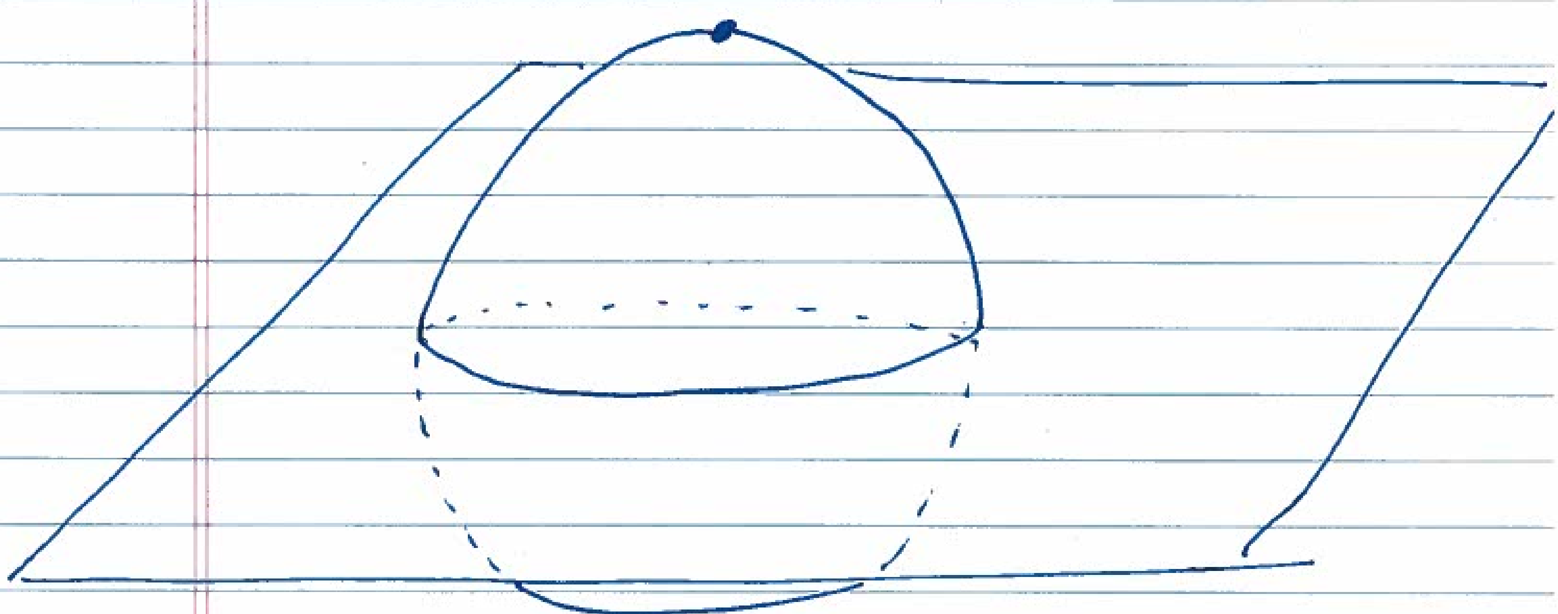
Each point z in the complex plane corresponds to ____________________:

Draw a

Conversely,

We let ∞ correspond to the point ___ and call $\mathbb{C} \cup \{\infty\}$ the _______________.

Neighborhood of ∞



Let $M>0$. A neighborhood of ∞ has the form

$$\{ \qquad\qquad \}.$$

Definition. Let $w_0 \in \mathbb{C}$. Suppose $f(z)$ is defined on a (deleted) neighborhood of ∞. We say $\lim_{z\to\infty} f(z) = w_0$ if

Let $z' = \frac{1}{z}$.

Hence

$$\lim_{z \to \infty} f(z) = w_0 \text{ iff}$$

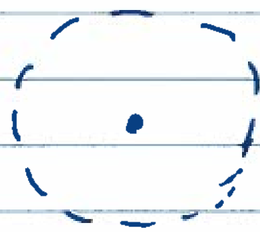
Example Find $\displaystyle\lim_{z \to \infty} \frac{iz+1}{z-i}$.

(p. 22)

Continuity

Let $z_0 \in \mathbb{C}$ and suppose $f(z)$ is a complex valued function defined on an open neighborhood of z_0,

ie



Def$^{\underline{n}}$ f is ~~continuous~~ continuous at z_0

if

(i)

(ii)

and (iii)

ie

Theorem Let $f(z) = u(x,y) + i\,v(x,y)$

where

Then f is continuous at $z_0 = x_0 + iy_0$ iff

Example $f(z) = (x + e^{xy}) - i \sin(xy)$ is

Also find $\lim\limits_{z \to \pi + i} f(z)$.

Derivatives

<u>Def^{n}</u> Let $z_0 \in \mathbb{C}$ and suppose $f(z)$ is a complex-valued function defined on some open neighborhood of z_0

We say $f(z)$ is <u>differentiable at z_0</u> if

If then we write

$$f'(z_0) =$$

<u>NOTE</u> Let $\Delta z :=$.

Then $z \to z_0$ iff $\Delta z \to$ ___

We can write

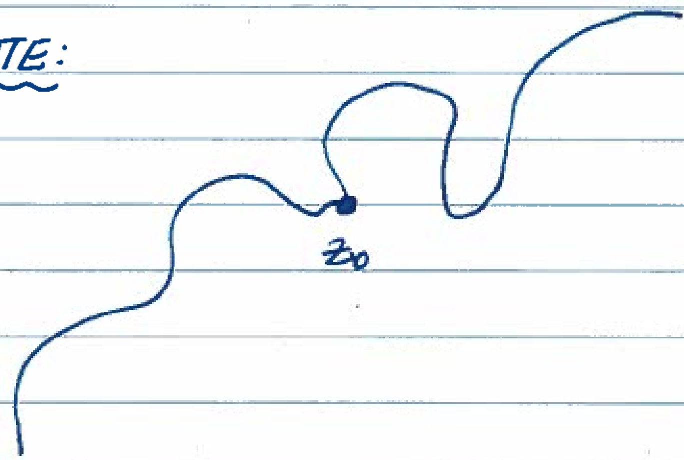
Hence f is differentiable at z iff

Example Show that $f(z) = z^2$ is differentiable at every z and find $f'(z)$.

Example Determine where $f(z) = \bar{z}$ is differentiable.

(p. 25A)

NOTE:



Differentiation Formulas

Notation $\frac{d}{dz} f(z) =$ (assuming

Suppose f, g are differentiable functions $f : \mathbb{C} \to \mathbb{C}$ & $g : \mathbb{C} \to \mathbb{C}$. Then

(1) $\frac{d}{dz} c =$ if

(2) $\frac{d}{dz} c f(z) =$ if

(3) $\frac{d}{dz} z^n =$ for

(4) $\frac{d}{dz} \left(f(z) + g(z) \right) =$

(5) $\frac{d}{dz} \left(f(z)\, g(z) \right) =$

(6) $\frac{d}{dt} f(g(z)) =$

<u>Example</u> Let $f(z) = (1+z+z^2)^{100}$

<u>Cauchy-Riemann Equations</u>

<u>Theorem</u> Suppose $f(z) = u(x,y) + iv(x,y)$ is a complex valued function defined on some _ _ _ _ _ _ _ _ _ of z_0, and suppose f is d'ble at $z_0 = x_0 + iy_0$. [Here $u, v \in$].
Then the

and satisfy the Cauchy-Riemann equations

at . Further,

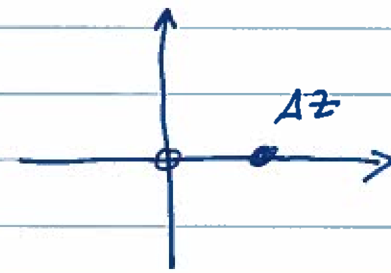
$f'(z_0) =$

PROOF: Suppose f is d'ble at $z_0 = x_0 + i y_0$. Then the limit

$$f'(z_0) =$$

exists. First we let $\Delta z \to 0$ along the real axis, so that

$$\Delta z =$$



Therefore,

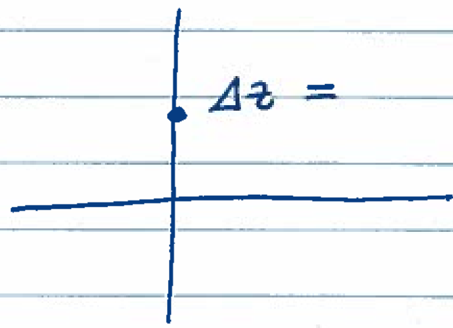
$$f'(z_0) = \lim_{\substack{\Delta z \to 0 \\ \text{(along real axis)}}}$$

$$=$$

Hence the partial derivatives must and

$$f'(z_0) =$$

Now let $\Delta z \to 0$ along imaginary axis.



So $f'(z_0) = \lim_{\substack{\Delta z \to 0 \\ \text{(along imaginary axis)}}}$

$=$

So the partial derivatives
must and

$$f'(z_0) =$$

Hence

ie $\square$

Corollary If the Cauchy-Riemann equations are
not satisfied at (x_0, y_0) then

<u>Example</u> Let $f(z) = |z|^2$.
What do the Cauchy-Riemann Equations imply about the differentiability of $f(z)$?

<u>WARNING</u>

SUFFICIENT CONDITIONS

Theorem

Suppose the function $f(z) = u(x,y) + i\,v(x,y)$ $(u, v \in \mathbb{R})$ is defined on an open neighborhood $D(z_0, r)$ of $z_0 = x_0 + iy_0$. Suppose the partial derivatives

all

and are . Then if the Cauchy-Riemann equations

hold at . Then

PROOF See pp. 66-67 of the Text.

Example Let $f(z) = |z|^2$. Does the Theorem apply? What does it imply?

<u>Example</u> Let $f(z) = 2xy - i(x^2 - y^2)$
Where is f d'ble? & find $f'(z)$.

The Cauchy-Riemann Equations in Polar Form

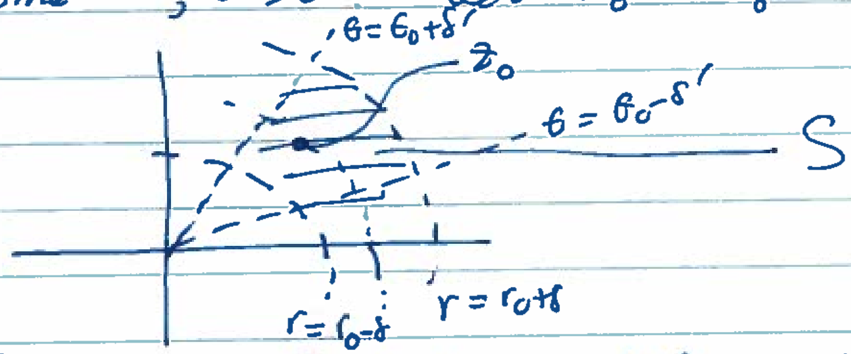
Theorem

Suppose

$$f(re^{i\theta}) = u(r,\theta) + iv(r,\theta)$$

for $0 < r_0 - \delta < r < r_0 + \delta$, $\theta_0 - \delta' < \theta < \theta_0 + \delta'$

some $\delta, \delta' > 0$. Let $z_0 = r_0 e^{i\theta_0}$



Suppose the first order partial derivatives

$$\frac{\partial u}{\partial r}, \frac{\partial u}{\partial \theta}, \frac{\partial v}{\partial r}, \frac{\partial v}{\partial \theta}$$

exist everywhere in S & are continuous at (r_0, θ_0).

If the Cauchy-Riemann Equations

hold at _ _ _ _ _, then f is d'ble at z_0

and

$$f'(z_0) =$$

Proposition

Let $f(z) = u(x,y) + iv(x,y)$
is defined on a ~~nei~~ neighborhood of the _ _ _ _ _
point $z_0 = x_0 + iy_0$. Suppose the partial
derivatives
exist at _ _ _ _ and the Cauchy-Riemann Equations

hold at _ _ _ _. Then the Cauchy-Riemann Equations
(polar form)

hold at _ _ _ _, where

$U(r,\theta) =$, $V(r,\theta) =$

PROOF We let

$U(r,\theta) = u(x,y)$, $V(r,\theta) = v(x,y)$
where $x =$, $y =$.
Then

$\frac{\partial U}{\partial r} =$

$\frac{\partial V}{\partial r} =$

$$\frac{\partial u}{\partial \theta} =$$

$$\frac{\partial v}{\partial \theta} =$$

Hence

$$\begin{pmatrix} \frac{\partial u}{\partial r} \\ \frac{\partial u}{\partial \theta} \end{pmatrix} = \begin{pmatrix} & \\ & \end{pmatrix} \begin{pmatrix} \\ \\ \end{pmatrix},$$

and

$$\begin{pmatrix} \frac{\partial v}{\partial r} \\ \frac{\partial v}{\partial \theta} \end{pmatrix} = \begin{pmatrix} & \\ & \end{pmatrix} \begin{pmatrix} \\ \\ \end{pmatrix}.$$

Let $A = \begin{pmatrix} & \\ & \end{pmatrix}.$

Then $\det A =$

so A is

If $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ & $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

then

$$\frac{\partial u}{\partial \theta}$$

and

$$\frac{1}{r}\frac{\partial u}{\partial \theta} = \qquad ,$$

which is . also if $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ & $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

then

$$\frac{\partial v}{\partial \theta} =$$

<u>Exercise</u> Assume $0 < r_0$.
Show that if

$$\frac{1}{r}\frac{\partial v}{\partial \theta} = \frac{\partial u}{\partial r}, \quad \& \quad \frac{1}{r}\frac{\partial u}{\partial \theta} = \frac{-\partial v}{\partial r}$$

hold at (r_0, θ_0), then the Cauchy-Riemann Equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial v}{\partial x} = \frac{-\partial u}{\partial y}$$ hold at

$$z_0 = r_0 e^{i\theta_0} = r_0 (\cos\theta_0 + i\sin\theta_0).$$

(p.39)

Exercise Let $z_0 = r_0 e^{i\theta_0}$ where $0 < r_0$.
Let $f(re^{i\theta}) = U(r,\theta) + iV(r,\theta)$
as in the Theorem. If f is d'ble at z_0 prove
that

$$f'(z_0) = e^{-i\theta_0}\left(\frac{\partial U}{\partial r}(r_0,\theta_0) + i\frac{\partial V}{\partial r}(r_0,\theta_0)\right).$$

(p.40)

Example

Let $D = \{ z \in \mathbb{C} : \ z \neq 0 \ \& \ -\pi < \arg z < \pi \}$

Define $g : D \to \mathbb{C}$ by $g(z) = \sqrt{r}\, e^{i\theta/2}$
where $z = re^{i\theta}$, $r > 0$ & $\theta = \operatorname{Arg} z$.

$g(z)$ is the _ _ _ _ _ _ _ value of $\sqrt{z}$.

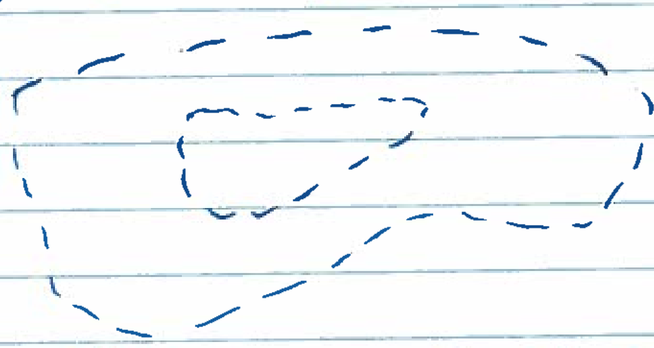
Show g is d'ble on D & find $g'(z)$.

(p. 41)

ANALYTIC FUNCTIONS

Definition Let $z_0 \in \mathbb{C}$ and suppose f is a complex-valued function defined on some open neighborhood of z_0. We say f is analytic at z_0 if

If f is defined a some open set S then we say f is analytic on S if



NOTES (1) A set $S \subset \mathbb{C}$ is open if

(2) Some books use

Definition: $f:\mathbb{C}\to\mathbb{C}$ is **entire** if $f(z)$ is ______________.

Remark:

Example Let $f:\mathbb{C}\to\mathbb{C}$ by $f(z)=|z|^2$.
We have seen that f is d'ble ______________.
f is ______________.
Hence f is ______________.

Example. Let $D=\mathbb{C}\setminus\{0\}=\{z\in\mathbb{C}: z\neq 0\}$.
Then D is an ______ subset of $\mathbb{C}$.
Let $f: D\to\mathbb{C}$ by $f(z)=\frac{1}{z}$.
Then f is ______________
on ______ and
$f'(z)=$

Notice that f is analytic on ______________
The point ______ is called a ______________
of f.

Definition If f is ______________
but ______________ then
z_0 is called ______________.

(p. 44)

<u>Example</u> Let $f : \mathbb{C} \to \mathbb{C}$ by $f(z) = \dfrac{1}{(z-1)(z-2)}$.

<u>Proposition</u> Suppose D is an ____ subset of $\mathbb{C}$.
Suppose $f : D \to \mathbb{C}$, $g : D \to \mathbb{C}$ are analytic functions.
Then

(1) $f(z) + g(z)$ is

(2) $f(z)\,g(z)$ is

(3) $\dfrac{f(z)}{g(z)}$ is

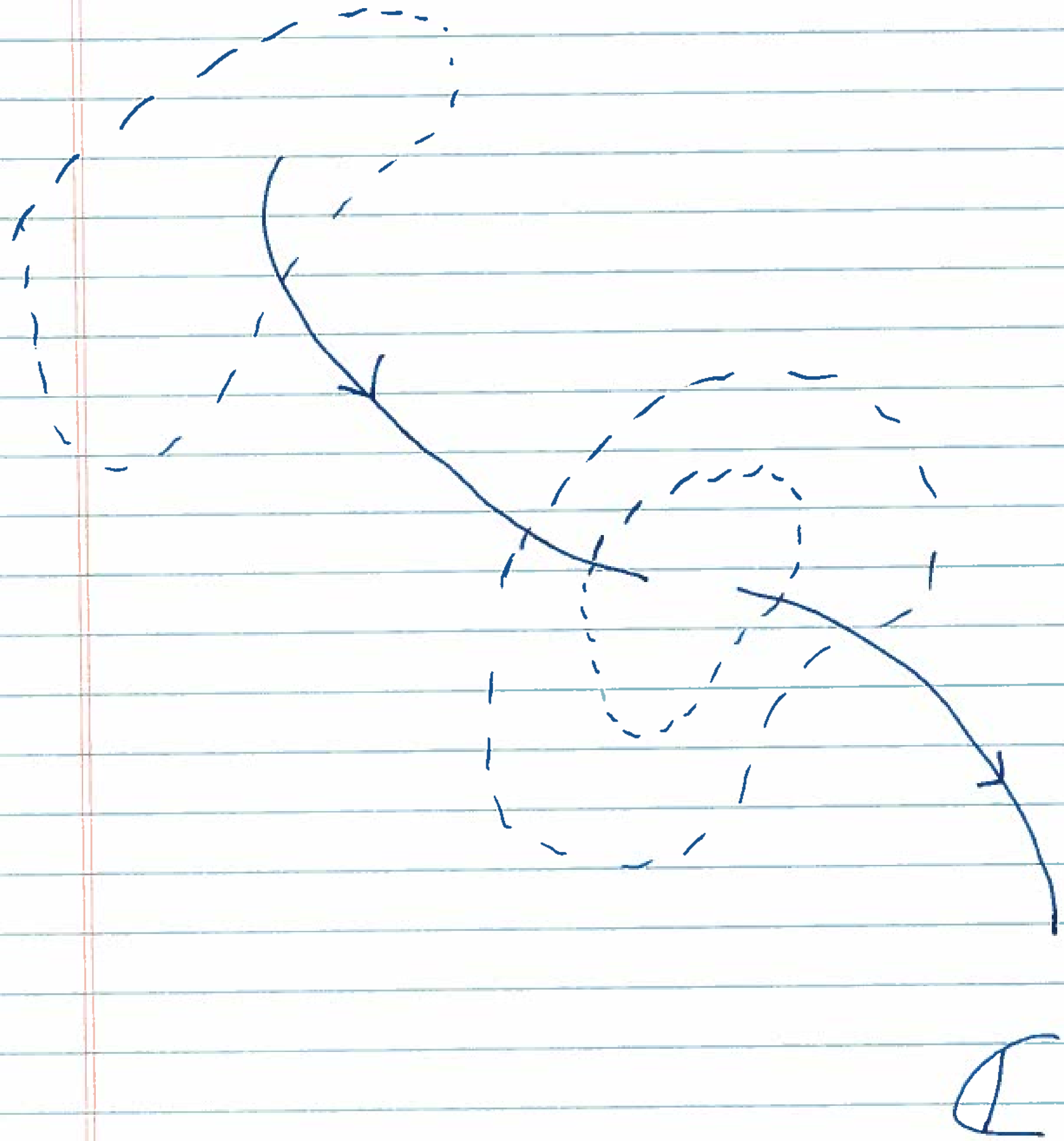
<u>Proposition</u> Suppose D, E are open subsets of $\mathbb{C}$,
$f : D \to \mathbb{C}$, $g : D \to \mathbb{C}$ are analytic
and ______, Then

$g \circ f :$ ___ $\to \mathbb{C}$ by $g \circ f(z) =$ ___.

is analytic on ___ and

$(g \circ f)'(z) =$

(p.44A)



Example (#5, p.76)

Let $g(z) = \sqrt{r}\, e^{i\theta/2}$ (for $r>0$, $-\pi < \theta < \pi$ where $z = re^{i\theta}$).

Then g is analytic on $D = \{ \qquad \}$ and

$g'(z) =$

Show $G(z) = g(2z-2+i)$ is analytic in the half-plane $x>1$ with derivative

$G'(z) =$

Let $h: \mathbb{C} \to \mathbb{C}$ by $h(z) = 2z-2+i$
then h is ____.

Let $\mathcal{H} = \{ z \in \mathbb{C} : \operatorname{Re}(z) > 1 \}$

We need to show that ____.

$$\xrightarrow{\quad h \quad}$$

Hence $G =$ ___ : ___ $\longrightarrow$ ___
is ____ and

$$G'(z) =$$

<u>Theorem</u> 1. Suppose $D \subset \mathbb{C}$ is a domain (i.e. an ___ ______ subset of $\mathbb{C}$). Suppose $f : D \longrightarrow \mathbb{C}$ is ____ and

$$f'(z) = 0 \quad \text{for } \_\_\_\_\_\_ .$$

Then

We need some results from Calculus:

Theorem 2

(i) Let $a < b$ be real constants.

If $f : (a, b) \longrightarrow \mathbb{R}$ is _____ and

$$f'(x) = 0 \quad \text{for } \_\_\_\_\_,$$

then

(ii) Suppose $f : [a, b] \longrightarrow \mathbb{R}$ is _____

and _____ on _____ and

$$f'(x) = 0 \quad \text{for } \_\_\_\_\_,$$

then

(iii) Let $a < x_0 < b$ and suppose

$f : (a, b) \longrightarrow \mathbb{R}$ is _____ at x_0

then f is _____ at x_0.

PROOF of Theorem 1

Suppose $D \subset \mathbb{C}$ is a domain & $f : D \longrightarrow \mathbb{C}$

is analytic and $f'(z) = 0$ for all $z \in D$.

Let $f(z) = u(x, y) + i\, v(x, y)$ $\quad (u, v \in \mathbb{R})$

for $z \in D$. Then

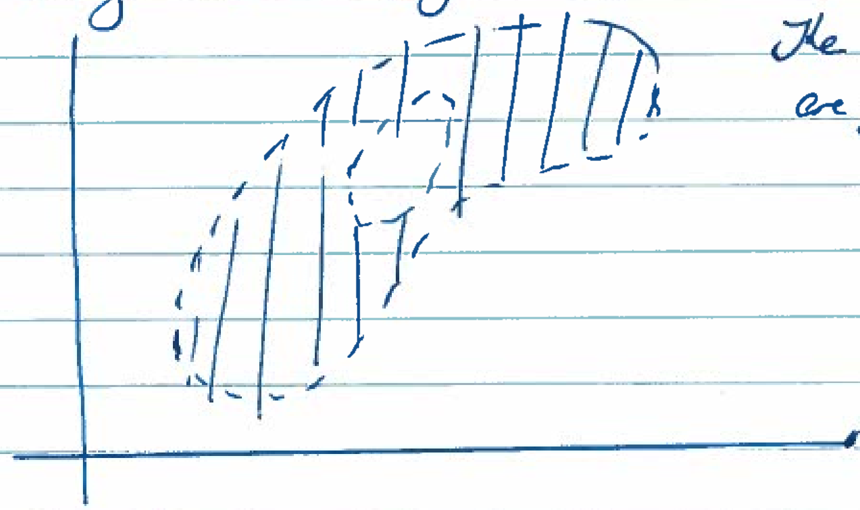
$$f'(z) = \qquad\qquad = \qquad\qquad = 0$$

for all $z \in D$ so that

$$\frac{\partial u}{\partial x} = \qquad , \quad \frac{\partial u}{\partial y} = \qquad , \quad \frac{\partial v}{\partial x} = \qquad , \quad \frac{\partial v}{\partial y} = \qquad ,$$

for _____.

(1) First we show that f is constant along any horizontal line segment $\ell \subset D$.



The points on ℓ are given by

Now for $z =$,

$f(z) =$

Let $g_1 :$ ___ $\rightarrow$ ___ by $g_1(x) =$

Then g_1 is ______ on ______ and

______ on ______ and

$g_1'(x) =$

for ______. since f is ______

and exist for

Hence

$g_1'(x) =$

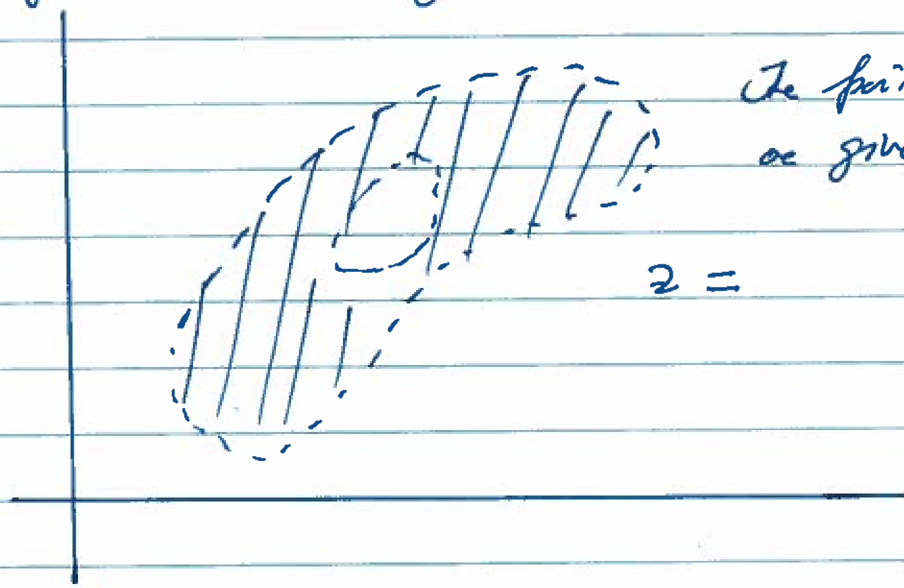
for & g_1 is a ______.

Hence there is a ___ constant k_1

such that

$=$

(2) Similarly we can show f is constant along any vertical line segment $\ell' \subset D$.



The points on ℓ' are given by

$z =$

Now for $z =$

$f(z) =$

Let $g_2 :$ ___ $\longrightarrow$ ___ by $g_2(x) =$

Then g_2 is ______ on ___ and

______ on ___ and

$g_2'(x) =$

for ______ since f is ______

and exist for

Hence

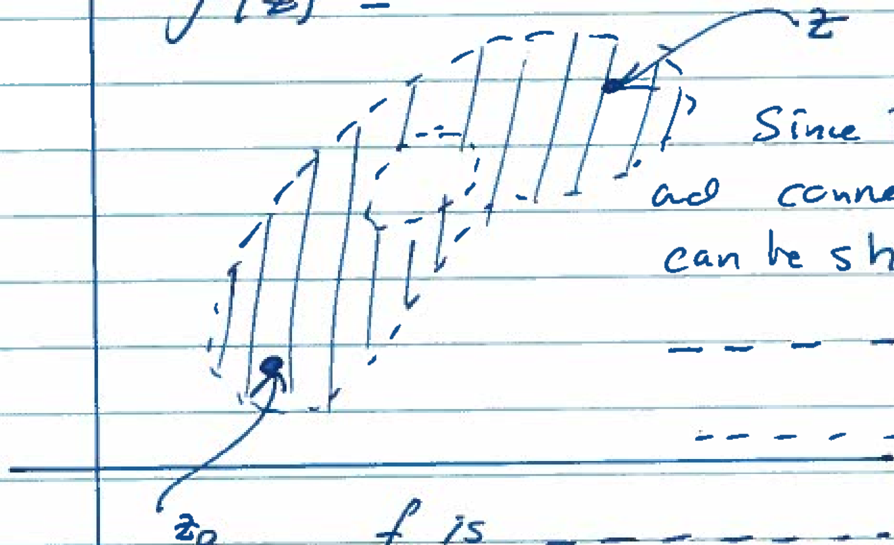
$g_2'(x) =$

for & g_2 is a ______.

any f is ______.

(3) Now fix any point $z_0 \in D$ & let z be any point in D. We show that

$f(z) =$



Since D is open and connected it can be shown that

_ _ _ _ _ _ _ _ _ _ _ _

_ _ _ _ _ _ _ _ _ _ _ _ _ . . .

f is _ _ _ _ _ _ _ _ _
along _ _ _ _ _ _ _ _ _ . . .

It follows that
$f(z) =$
for _ _ _ _ _ . . Therefore

f is _ _ _ _ _ _ _ _ _ _ _ . $\square$

<u>Example</u> (See p. 75 of text)
Suppose that a function

$$f(z) = u(x,y) + i\, v(x,y)$$

and its conjugate

$$\overline{f(z)} = u(x,y) - i\, v(x,y)$$

are both analytic on a domain D (i.e. $D \subset \mathbb{C}$
and D is __ __ __ __ and __ __ __ __ __ __)
Show that $f(z)$ is a constant function on D.

<u>Example</u> (See pp. 75–76 of text)
Suppose $f(z)$ is analytic on a domain D and $|f(z)|$ is constant on D. Prove that $f(z)$ is a constant function on D.

Harmonic Functions

Definition: Let $D \subset \mathbb{R}^2$ be a domain (ie D is ______ and _______). A function $h: D \to \mathbb{R}$ is harmonic if

Example Let $h(x,y) = x^3 - 3xy^2$.

Theorem Let $D \subset \mathbb{C}$ be a domain and suppose
$$f : D \to \mathbb{C}$$
is analytic. If $f(z) = u(x,y) + i\,v(x,y)$
then the functions $u(x,y)$ and $v(x,y)$ are — — — —
— — — — —.

Proof: Suppose f is analytic. Then

<u>Example</u> The function $f(z) = z^3$ is analytic (in fact ______).

$f(x+iy) =$