Chapter 2. Analytic Functions

Functions of a Complex Variable

We will consider functions \( f : S \rightarrow \mathbb{C} \)
where \( S \subset \mathbb{C} \) is the domain of \( f \).

Examples: Find the domains of the following functions:

(i) \( f(z) = \frac{1}{z} \)  
(ii) \( f(z) = \text{Arg}(z) \)  
(iii) \( f(z) = \text{Arg}(\frac{1}{z}) \).

Note: Let \( S \subset \mathbb{C} \). Any function \( f : S \rightarrow \mathbb{C} \)
can be written as:

\[
f(z) = u + iv,
\]

where \( z = x + iy \in S \).

The functions \( u \) and \( v \) are:

where
Example. Write the function $f(z) = \frac{1}{z}$ in the form $f(z) = u(x,y) + i v(x,y)$.

The mapping $w = z^2$

<table>
<thead>
<tr>
<th>$z$</th>
<th>$f(z) = z^2$</th>
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<tbody>
<tr>
<td>$2i$</td>
<td>$-4$</td>
</tr>
<tr>
<td>$1+i$</td>
<td>$2+i$</td>
</tr>
<tr>
<td>$2$</td>
<td>$4$</td>
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Ex. Let $f(z) = z^2$.

(i) Find the image of the imaginary axis
(ii) Find the image of any vertical line $\text{Re}(z) = c$

where $c$ is a non-zero constant. Sketch the image.
(iii) Find the image of the real axis.

(iv) Find the image of any horizontal line \( \text{Im}(z) = c \) where \( c \) is any nonzero constant. Sketch the image.

**Note:** Let \( SC \subset \mathbb{C} \) and suppose \( f : S \to \mathbb{C} \). Let \( A \subset S \). The image of \( A \) under \( f \) is denoted by \( f(A) \) and is defined by

Let \( f : \mathbb{C} \to \mathbb{C} \) by \( f(z) = z^2 \). Then

\[
\begin{align*}
f(z) &= (x + iy)^2 \\
&= u(x, y) + iv(x, y)
\end{align*}
\]

where \( u(x, y) = \) and \( v(x, y) = \)

(v) Let \( z = \) where

Then \( f(z) = \) where the image of the imaginary axis under \( f \) is \( i \).
(iii) let \( z = \) 
\[
\text{then } f(z) =
\]
(iii) Let \( z = \) 
\[ \theta \in \theta \]
Then \( f(\theta) = \)

(iv) Let \( z = \)
\[ \text{where} \]
\[ \text{then } f(z) = \]
1. $f(z) = z^2$
2. \( f(z) = \frac{1}{z} \)
Example: Find the image of the sector $0 \leq \theta \leq \frac{\pi}{4}$, $0 < r \leq 1$ under the map $f(z) = \frac{1}{z}$.

Let $z \in S$ then

$$w = \frac{1}{z}$$

where

Then $w = f(z) =$
Example: Let $f: \mathbb{C} \to \mathbb{C}$ by $f(z) = z^3$.
Describe this map.

Exercise
Under the map $w = \frac{1}{z}$, show that
(i) The image of any horizontal line (except the 0) is a circle.
(ii) The image of any vertical line (except the 0) is a circle.
Hint: Try to get $u^2 + v^2$ in terms of $u$ or $v$. 
**Limits**

**Definition:** Let $z_0 \in \mathbb{C}$. A deleted (punctured) neighborhood of $z_0$ has the form

$$D'(z_0, r) = \{ z \in \mathbb{C} : \}$$

**Definition:** Let $z_0 \in \mathbb{C}$ & suppose $f: D'(z_0, r) \to \mathbb{C}$ for some $r > 0$. Let $w_0 \in \mathbb{C}$. We say the limit of $f(z)$ as $z$ approaches $z_0$ is $w_0$ and write

if

![Diagram of deleted neighborhoods with points and arrows indicating direction and distance](diagram.jpg)
Example Using the definition show that
\[
\lim_{z \to i} (2z+1) = 2i+1.
\]

**Working:** Let \( \varepsilon > 0 \). We want to show that there is a \( \delta > 0 \) such that

\[
|f(z)| < \varepsilon \quad \text{if} \quad |z - i| < \delta.
\]

**Proof:** Let \( \varepsilon > 0 \) be any fixed positive real number. Suppose
Example: Use the definition to prove that

$$\lim_{z \to 1+i} \frac{i}{z} =$$

Working:
Proof: Let $\varepsilon > 0$ be any fixed positive real number.
Let $\delta =$
Theorems on Limits

Theorem Let \( f \) be defined on an deleted neighborhood of \( z_0 = x_0 + iy_0 \) & let \( w_0 = u + iv_0 \) abide \( x_0, y_0, u, v_0 \in \mathbb{R} \) & 

\[ f(z) = u(x, y) + iv(x, y) \]

for \( z = x + iy \) in domain of \( f \). Then

\[ \lim_{z \to z_0} f(z) = w_0 \]

if and only if

Example Let \( f(z) = \frac{1}{z} \).

Find \( \lim_{z \to 1+i} f(z) \).
Proof of Theorem

(⇒) Suppose \( \lim_{z \to z_0} f(z) = w_0 \).
Theorem

Suppose \( f(z) \) and \( g(z) \) are complex-valued functions defined on a deleted neighborhood of \( z_0 \), and suppose

\[
\lim_{z \to z_0} f(z) = w_0 \quad \text{and} \quad \lim_{z \to z_0} g(z) = w_1.
\]

Then

(1) \[
\lim_{z \to z_0} \left( f(z) + g(z) \right) =
\]

(2) \[
\lim_{z \to z_0} f(z) g(z) =
\]

(3) \[
\lim_{z \to z_0} \frac{f(z)}{g(z)} = \quad \text{if} \quad g(z) \neq 0
\]

(4) \[
\lim_{z \to z_0} c f(z) = \quad \text{if} \quad c \neq 0
\]

(5) \[
\lim_{z \to z_0} z^n = \quad \text{if} \quad c \neq 0
\]

(6) \[
\lim_{z \to z_0} P(z) =
\]

if \( P(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_0 \) is a polynomial.

(7) \[
\lim_{z \to z_0} \frac{P(z)}{Q(z)} = \quad \text{if} \quad \frac{P(z)}{Q(z)} \text{ is a rational function}
\]

and
THE POINT AT INFINITY

Suppose \( z_0 \in \mathbb{C} \) and \( f(z) \) is defined on a deleted neighborhood of \( z_0 \).

**Definition.** We say \( f(z) \) approaches \( \infty \) as \( z \to z_0 \) if

\[
\lim_{z \to z_0} f(z) = \infty
\]

**Note:** This holds iff

Here \( \lim_{z \to z_0} f(z) = \infty \) iff
The Riemann Sphere

Each point \( z \) in the complex plane \( \mathbb{C} \) corresponds to ---

Draw a

Conversely,

We let \( \infty \) correspond to the point --- and call \( \{ z \mid z \in \mathbb{C} \} \) the ---
Let $M > 0$. A neighborhood of $\infty$ has the form
\[ \{ x \in \mathbb{R} | x > M \} \cup \{ x \in \mathbb{R} | x < -M \}. \]

Definition. Let $w \in \mathbb{C}$. Suppose $f(z)$ is defined on a (deleted) neighborhood of $\infty$. We say $\lim_{z \to \infty} f(z) = w$ if
\[ \text{let } z' = \frac{1}{z}. \]
Hence
\[ \lim_{z \to \infty} f(z) = w_0 \text{ iff } \]

**Example** Find \[ \lim_{z \to \infty} \frac{iz + 1}{z - i} \]
Continuity

Let \( z_0 \in \mathbb{C} \) and suppose \( f(z) \) is a complex-valued function defined on an open neighborhood of \( z_0 \), i.e.

\[
\begin{array}{c}
\text{Def:} \quad f \text{ is continuous at } z_0 \\
\text{if} \\
(i) \\
(ii) \\
(iii) \\
\text{ie}
\end{array}
\]

Theorem. Let \( f(z) = u(x,y) + iv(x,y) \) where \( f \) is continuous at \( z_0 = x_0 + iy_0 \) iff

\[
\text{Example} \quad f(z) = (x + e^{2y}) - i \sin(2y) \text{ is}
\]
Also find \( \lim_{z \to \pi + i} f(z) \).

**Derivatives**

**Definition**

Let \( z_0 \in \mathbb{C} \) and suppose \( f(z) \) is a complex-valued function defined on some open neighborhood of \( z_0 \).

If

\[
\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = L
\]

we say \( f(z) \) is **differentiable at** \( z_0 \).

If

\[
f'(z_0) = \]

**Note**

Let \( \Delta z := \)

\[
\lim_{z \to z_0} \text{ iff } \Delta z \to \_ \_ \_ \]

We can write
Hence \( f \) is differentiable at \( z \) iff

Example Show that \( f(z) = z^2 \) is differentiable at every \( z \) and find \( f'(z) \).

Example Determine where \( f(z) = \overline{z} \) is differentiable.
Differentiation Formulas

Notation \( \frac{d}{dz} f(z) = \) (assuming

Suppose \( f, g \) are differentiable functions \( f: \mathbb{C} \to \mathbb{C} \) and \( g: \mathbb{C} \to \mathbb{C} \). Then

1. \( \frac{d}{dz} c = 0 \) if

2. \( \frac{d}{dz} c f(z) = c f'(z) \) if

3. \( \frac{d}{dz} z^n = n z^{n-1} \) if

4. \( \frac{d}{dz} (f(z) + g(z)) = f'(z) + g'(z) \)

5. \( \frac{d}{dz} (f(z) g(z)) = f(z) g'(z) + f'(z) g(z) \)
(6) \[ \frac{d}{dt} f(g(t)) = \]

Example. Let \( f(z) = (1 + z + z^2)^{100} \)

**Cauchy–Riemann Equations**

**Theorem.** Suppose \( f(z) = u(x,y) + iv(x,y) \) is a complex-valued function defined on some

--- of \( z_0 \), and suppose \( f \) is
differentiable at \( z_0 = x_0 + iy_0 \). [i.e., \( u, v \in C \).]

Then the

and satisfy the Cauchy–Riemann equations

at.

Further,

\[ f'(z_0) = \]
Proof: Suppose \( f \) is differentiable at \( z_0 = x_0 + i y_0 \). Then the limit \( f'(z_0) = \) exists. First we let \( \Delta z \to 0 \) along the real axis, so that \( \Delta z = \) 

Therefore,

\[
    f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} \quad \text{(along real axis)}
\]
Hence the partial derivatives must and

\[ f'(z_0) = \]

Now let \( \Delta z \to 0 \) along imaginary axis.

\[ \Delta z = \]

So \( \lim_{\Delta z \to 0} f'(z_0) = \) (along imaginary axis)
So the partial derivatives must and

\[ f'(z_0) = \]

Hence

\[ \text{Corollary If the Cauchy-Riemann equations are not satisfied at } (x_0, y_0) \text{ then} \]
Example: Let $f(z) = |z|^2$.
What do the Cauchy-Riemann Equations imply about the differentiability of $f(z)$?

**WARNING**
**Sufficient Conditions**

**Theorem**

Suppose the function \( f(z) = u(x,y) + iv(x,y) \) \((u,v \in \mathbb{R})\)

is defined on an open neighborhood \( D(z_0, r) \) of \( z_0 = x_0 + iy_0 \). Suppose the partial derivatives of all and one equations hold at \( z_0 \). Then if the Cauchy-Riemann equations hold at \( z_0 \), then

**Proof**

See pp. 66-67 of the Text.

**Example**

Let \( f(z) = |z|^2 \). Does the theorem apply? What does it imply?
Example: Let \( f(z) = 2xy - i(x^2 - y^2) \). What is \( f \) differentiable? Find \( f'(z) \).
The Cauchy-Riemann Equations in Polar Form

**Theorem**

Suppose

\[ f(re^{i\theta}) = u(r, \theta) + iv(r, \theta) \]

for \( 0 < \delta < r < r_0 + \delta, \theta_0 - \delta' < \theta < \theta_0 + \delta' \)

some \( \delta, \delta' > 0 \). Let \( z_0 = r_0 e^{i\theta_0} \)

Suppose the first order partial derivatives

\[ \frac{\partial u}{\partial r}, \frac{\partial u}{\partial \theta}, \frac{\partial v}{\partial r}, \frac{\partial v}{\partial \theta} \]

exist everywhere in \( S \) & are continuous at \( (r_0, \theta_0) \).

If the Cauchy-Riemann Equations

\[ \frac{\partial u}{\partial r} = \frac{\partial v}{\partial \theta}, \quad \frac{\partial u}{\partial \theta} = -\frac{\partial v}{\partial r} \]

hold at \( z_0 \), then \( f \) is differentiable at \( z_0 \) and

\[ f'(z_0) = \]
Proposition

Let \( f(z) = u(x, y) + iv(x, y) \) is defined on a neighborhood of the \( \ldots \) point \( z_0 = x_0 + iy_0 \). Suppose the partial derivatives exist at \( \ldots \) and the Cauchy-Riemann Equations hold at \( \ldots \). Then the Cauchy-Riemann Equations (polar form)

\[
\begin{align*}
U(r, \theta) &= \ldots, \\
V(r, \theta) &= \ldots
\end{align*}
\]

Proof We let

\[
\begin{align*}
U(r, \theta) &= u(x, y), \\
V(r, \theta) &= v(x, y)
\end{align*}
\]

where \( x = \ldots \), \( y = \ldots \).

Then

\[
\begin{align*}
\frac{\partial u}{\partial x} &= \ldots, \\
\frac{\partial v}{\partial r} &= \ldots
\end{align*}
\]
\[ \frac{\partial u}{\partial \theta} \]

\[ \frac{\partial v}{\partial \theta} \]

Here

\[
\begin{pmatrix}
\frac{\partial u}{\partial r} \\
\frac{\partial u}{\partial \theta}
\end{pmatrix}
= \begin{pmatrix}
\text{ } \\
\text{ }
\end{pmatrix}
\]

and

\[
\begin{pmatrix}
\frac{\partial v}{\partial r} \\
\frac{\partial v}{\partial \theta}
\end{pmatrix}
= \begin{pmatrix}
\text{ } \\
\text{ }
\end{pmatrix}
\]

Let

\[ A = \begin{pmatrix}
\text{ }
\end{pmatrix} \]

Then det \( A = \)

so \( A \) is
If \( \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \) & \( \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \)

Then

\[ \frac{\partial u}{\partial \theta} \]

And

\[ \frac{1}{r} \frac{\partial u}{\partial \theta} = \]  

which is  

As if \( \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \) & \( \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \)

Then

\[ \frac{\partial V}{\partial \theta} = \]
Exercise 3 Assume $0 < r_0$.
Show that if

$$
\frac{1}{r} \frac{\partial v}{\partial \theta} = \frac{\partial u}{\partial r}, \quad \frac{1}{r} \frac{\partial u}{\partial \theta} = \frac{\partial v}{\partial r}
$$

hold at $(r_0, \theta_0)$. Then the Cauchy-Riemann Equations

$$
\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}
$$

hold at

$$
\theta_0 = r_0 e^{i \theta_0} = r_0 (\cos \theta_0 + i \sin \theta_0).
$$
Exercise. Let \( z_0 = r_0 e^{i\theta_0} \), where \( 0 < r_0 < R \).

Let \( f(r e^{i\theta}) = U(r, \theta) + iV(r, \theta) \)
as in the theorem. If \( f \) is differentiable at \( z_0 \), prove that

\[
f'(z_0) = e^{-i\theta_0} \left( \frac{\partial U}{\partial r}(r_0, \theta_0) + i \frac{\partial V}{\partial r}(r_0, \theta_0) \right).
\]
Example

Let \( D = \{ z \in \mathbb{C} : z \neq 0 \text{ and } -\pi < \arg z < \pi \} \)

Define \( g : D \to \mathbb{C} \) by \( g(z) = \sqrt{r} e^{i \theta / 2} \)

where \( z = r e^{i \theta} \), \( r > 0 \) and \( \theta = \arg z \).

\( g(z) \) is the principal value of \( \sqrt{z} \).

Show \( g \) is differentiable and find \( g'(z) \).
**Analytic Functions**

Definition: Let \( z_0 \in \mathbb{C} \) and suppose \( f \) is a complex-valued function defined on some open neighborhood of \( z_0 \). We say \( f \) is analytic at \( z_0 \) if

\[
\]

If \( f \) is defined on an open set \( S \) then we say \( f \) is analytic on \( S \) if

\[
\]

**Note:**

1. A set \( S \subset \mathbb{C} \) is open if

2. Some books use
Definition: \( f : \mathbb{C} \to \mathbb{C} \) is entire if \( f(z) \) is \( \overline{\text{a}} \) \( \overline{\text{b}} \).

Remark:

Example. Let \( f : \mathbb{C} \to \mathbb{C} \) by \( f(z) = 1 \cdot z^2 \).
We have seen that \( f \) is \( \overline{\text{d}} \) \( \overline{\text{e}} \),
\( f \) is \( \overline{\text{f}} \).
More \( f \) is \( \overline{\text{g}} \).

Example. Let \( D = \mathbb{C} \setminus \{0\} = \{ z \in \mathbb{C} : z \neq 0 \} \).
Then \( D \) is an \( \overline{\text{h}} \) subset of \( \mathbb{C} \).
Let \( f : D \to \mathbb{C} \) by \( f(z) = \frac{1}{z} \).
Then \( f \) is \( \overline{\text{i}} \) on \( \overline{\text{j}} \) and
\( f'(z) = \overline{\text{k}} \).

Notice that \( f \) is analytic on \( \overline{\text{l}} \).
The point \( \overline{\text{m}} \) is called \( \overline{\text{n}} \) of \( f \).

Definition. If \( f \) is \( \overline{\text{o}} \) but \( \overline{\text{p}} \),
\( z_0 \) is called \( \overline{\text{q}} \).
Example: Let \( f : \mathbb{C} \rightarrow \mathbb{C} \) by \( f(z) = \frac{1}{(z-1)(z-2)} \).

Proposition: Suppose \( D \) is an open subset of \( \mathbb{C} \). Suppose \( f : D \rightarrow \mathbb{C} \), \( g : D \rightarrow \mathbb{C} \) are analytic functions. Then

1. \( f(z) + g(z) \) is
2. \( f(z) \cdot g(z) \) is
3. \( \frac{f(z)}{g(z)} \) is

Proposition: Suppose \( D, E \) are open subsets of \( \mathbb{C} \), \( f : D \rightarrow \mathbb{C} \), \( g : D \rightarrow \math{E} \) are analytic and \( \cdots \). Then

\( g \circ f : \cdots \rightarrow \mathbb{C} \) by \( (g \circ f)(z) = \cdots \)
is analytic on \( \cdots \) and

\( (g \circ f)'(z) = \)
Example (§5, p. 76)

Let \( g(z) = \sqrt{r} \, e^{i\theta} \) (for \( r > 0 \), \(-\pi < \theta < \pi \)).

Then \( g \) is analytic on \( D = \{ z : \Re(z) > 0, \, \Re(z) < \pi \} \).

Also, \( g'(z) = \frac{i}{\sqrt{r}} e^{i\theta} \).

Show that \( G(z) = g(2z - 2 + i) \) is analytic in the half-plane \( \Re(z) > 1 \) with derivative \( G'(z) = \).

Let \( h: \mathbb{C} \rightarrow \mathbb{C} \) by \( h(z) = 2z - 2 + i \).

Then \( h \) is -

Let \( H = \{ z \in \mathbb{C} : \Re(z) > 1 \} \)

We need to show that -
$h$

Hence $G = \quad \rightarrow$ is \quad and

$G'(x) =$

**Theorem 1.** Suppose $D \subseteq \mathbb{C}$ is a domain (i.e., an \quad subset of $\mathbb{C}$). Suppose $f: D \rightarrow \mathbb{C}$ is \quad and

$f'(z) = 0$ for \quad.
We need some results from Calculus:

**Theorem 2**

(i) Let $a < b$ be real constants.

If $f : (a, b) \to \mathbb{R}$ is _____ and

$f'(x) = 0$ for _____

then

(ii) Suppose $f : [a, b] \to \mathbb{R}$ is _____ and _____ on _____ and

$f'(x) = 0$ for _____

then

(iii) Let $a < x_0 < b$ and suppose

$f : (a, b) \to \mathbb{R}$ is _____ at $x_0$

and $f$ is _____ at $x_0$.

**Proof of Theorem**

Suppose $D \subseteq \mathbb{C}$ is a domain & $f : D \to \mathbb{C}$ is analytic and $f'(z) = 0$ for all $z \in D$.

Let $f(z) = u(x, y) + iv(x, y)$ \hspace{1cm} $(u, v \in \mathbb{R})$

for $z \in D$. Then

$f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial y} = 0$

for all $z \in D$. So $u$, $v$ are harmonic,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$
(1) First we show that $f$ is constant along any horizontal line segment $l \subset D$.

Let $g_1 : x \rightarrow y$ be given by $g_1(x) = \ldots$

Then $g_1$ is $\ldots$ on $\ldots$ and $\ldots$ on $\ldots$ and $g_1'(x) = \ldots$

So $\ldots$ since $f$ is $\ldots$

Hence $g_1'(x) = \ldots$

For $\ldots$ $g_1$ is a $\ldots$

Hence $\ldots$ is a $\ldots$ constant.
(2) Similarly we can show $f$ is constant along any vertical line segment $C \subset D$.

Let points $a, a'$ be given by

Now for $z = f(z) =$

Let $g_z : \mathbb{R} \to \mathbb{R}$ by $g_z(x) =$

Then $g_z$ is \( \ldots \) on \( \ldots \) and \( \ldots \) on \( \ldots \) and

\[ g_z'(x) = \]

For \( \ldots \) since $f$ is \( \ldots \) and \( \ldots \) exist for

Hence

\[ g_z'(x) = \]

For \( \ldots \) and $g_z$ is a \( \ldots \)

any $f$ is \( \ldots \)
(3) Now fix any point $z_0 \in D$ and let $z$ be any point in $D$. We show that $f(z) = z$.

Since $D$ is open and connected, it can be shown that $f(z_0) = z_0$.

It follows that $f(z) = z$ for all $z \in D$. Therefore, $f$ is constant. $\square$
Example (See p. 75 of text)

Suppose that a function
\[ f(z) = u(x, y) + i v(x, y) \]
and its conjugate
\[ \overline{f(z)} = u(x, y) - i v(x, y) \]
are both analytic on a domain \( D \) (i.e., \( D \subset \mathbb{C} \)) and \( D \) is \( \cdots \) and \( \cdots \)

Show that \( f(z) \) is a constant function on \( D \).
Example (see pp. 75-76 of text)
Suppose \( f(z) \) is analytic on a domain \( D \) and \( |f(z)| \) is constant on \( D \). Prove that \( f(z) \) is a constant function on \( D \).
Harmonic Functions

Definition: Let \( D \subseteq \mathbb{R}^2 \) be a domain (i.e. \( D \) is a region and \( \partial D \)). A function \( h : D \rightarrow \mathbb{R} \) is harmonic if
Example Let \( h(x, y) = x^3 - 3x(y^2) \).

Theorem Let \( D \subseteq \mathbb{C} \) be a domain and suppose \( f : D \to \mathbb{C} \) is analytic. If \( f(z) = u(x,y) + iv(x,y) \) then the functions \( u(x,y) \) and \( v(x,y) \) are ______.

Proof: Suppose \( f \) is analytic. Then ______.
Example: The function \( f(z) = z^3 \) is analytic (in fact \( \infty \)).

\[ f(x+iy) = \]