Chapter 3 – Elementary Functions

The Exponential Function

Definition The exponential function ____ is defined by

$$\exp(z) = e^z :=$$

for

NOTE: Let $z = x+iy$ $(x, y \in \mathbb{R})$. Then

$$\exp(z) = e^z =$$

Properties

(1) The exponential function is

(2) $\frac{d}{dz} e^z =$ ....

(3) $\exp(z_1 + z_2) = e^{(z_1+z_2)} =$ .... for $z_1, z_2 \in \mathbb{C}$.

(4) $\exp(z) = e^z$ has period ___
ie $\exp(z +\quad) =$

(5) $e^{z_1 - z_2} =$ for $z_1, z_2 \in \mathbb{C}$.

6) $e^z \neq$ ___
for ....

(7) The Range of the function $\exp(z)$ is ____
(the set of __ _________).

Proof:

(2), (1) Let $z = x+iy$ $(x, y \in \mathbb{R})$. Then

$$\exp(z) = e^{z} =$$

(3) Let $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$ where $x_1, y_1, x_2, y_2 \in \mathbb{R}$.
Then

$$\exp(z_1 + z_2) =$$

$$=$$

$$=$$

$$=$$

(6) Let $z \in \mathbb{C}$. Then $|\exp(z)| =$

(7) Let w be any non-zero complex number.
We want show that there is a $z \in \mathbb{C}$ such that

$$\exp(z) = w.$$

Let

$$w = \rho e^{i\varphi}$$

be the polar form of w, i.e. $|w| = \_\_\_$ &

$\_\_\_\_\_\_\_\_$.

Let $z = x + iy$, where $x, y \in \mathbb{R}$.

$|\exp(z)| =$

$$\exp(z) = e^{z} = w$$

NOTE: Let $w = \rho e^{i\varphi}$ where $\rho > 0$ & $\varphi \in \mathbb{R}$.

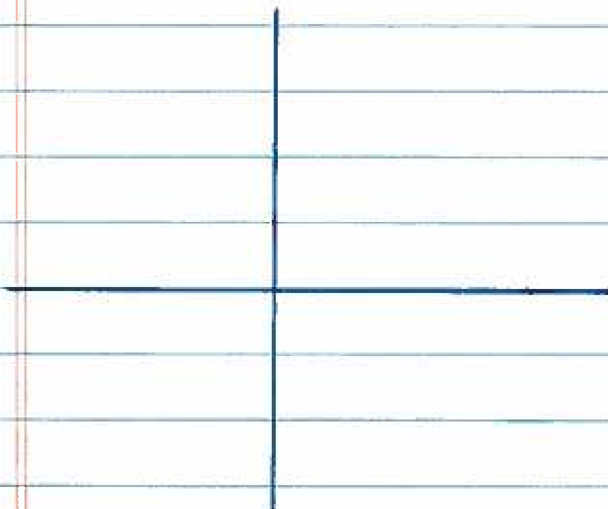
$\exp(z) = w$

if & only if

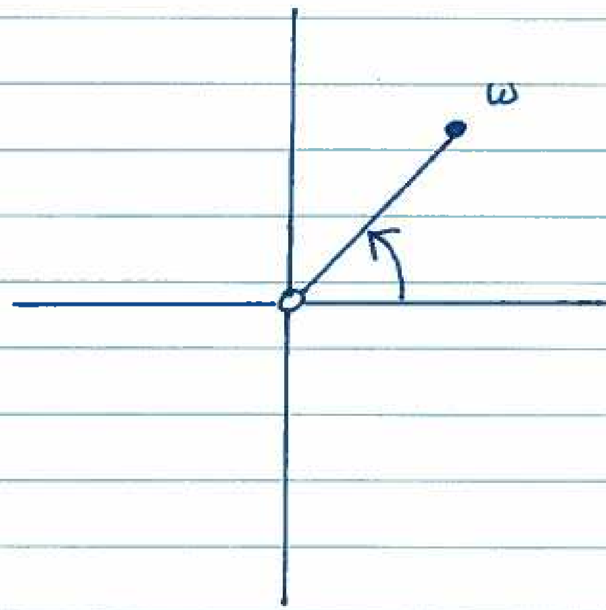
$z = \ldots\ldots\ldots\ldots\ldots.$

where $\ldots\ldots$

The exponential map



$\mathbb{C}$



$\mathbb{C} \setminus \{0\}$

Example : Find $\exp\left(\ln 2 + i\pi/4\right)$

Example Prove

$$|\exp(-2z)| < 1 \text{ iff } \operatorname{Re}(z) > 0$$

(for $z \in \mathbb{C}$).

PROOF:

Example Solve

$$e^{z} = -2.$$

The Logarithmic Function & Its Branches

Let $w = \rho e^{i\varphi}$ be a non-zero complex number (_ _ _ _ _.)
We have seen that the equation

$$e^z = w$$

has _ _ _ _ _ _ _ _ _ _ _ solutions

$$z =$$

We say each such z is a _ _ _ _ _ _ _ of _ _ _.
We write

$$\log w =$$

This is an example of a _ _ _ _ _ _ _ function, and is not a _ _ _ _ _ in the _ _ _ _ sense.

Definition Let $w \neq$ _ _ _. The principal value of the logarithm of w is defined by

$$\_\_\_ \; w := \ldots\ldots\ldots\ldots$$

NOTE: ...... $\operatorname{Arg} w$ ......

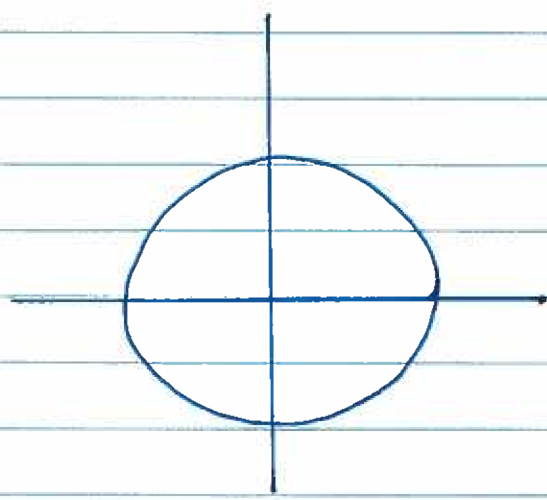
Example. Find

(1) $\operatorname{Log} 1$

(2) $\operatorname{Log} i$

(3) $\operatorname{Log}(-1)$

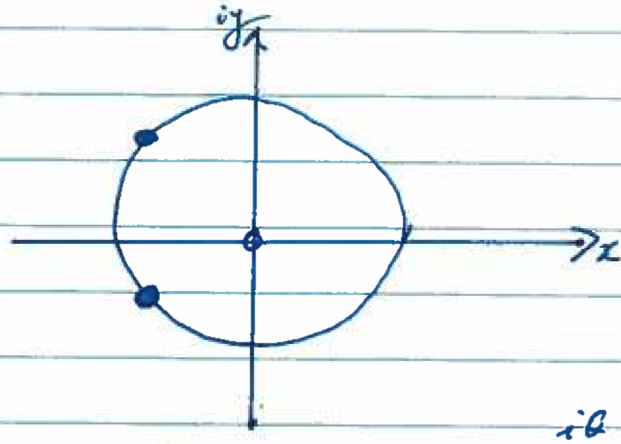
(4) $\operatorname{Log}(-1-i)$



NOTE: If $\operatorname{Log} z$ is defined all $z \neq 0$
then it is ________________ .

Let $z = e^{i\theta}$. Then

$$\lim_{\theta \to \pi^-} e^{i\theta}. \qquad \qquad \lim_{\theta \to \pi^+} e^{i\theta}$$



For $0 < \theta < \pi$, $\operatorname{Arg} e^{i\theta} =$ ___ .

So

$$\lim_{\theta \to \pi^-} \operatorname{Log}(e^{i\theta}) =$$

$$=$$

For $\pi < \theta < 2\pi$, $-\pi <$ < 0, &

$\operatorname{Arg} e^{i\theta} =$ ____ .

So

$$\lim_{\theta \to \pi^+} \operatorname{Log}(e^{i\theta}) =$$

$$=$$

$$=$$

So as $z \to -1$ along top semi-circle $\operatorname{Log}(z) \to$ ___

as $z \to -1$ along both semi-circle $\operatorname{Log}(z) \to$ __

Therefore $\lim_{z \to -1} \operatorname{Log}(z)$ _____

and $\operatorname{Log}(z)$ is ________ .

Theorem :
Let $D = \{z : \_\_\_\_\_ \text{ and } \_\_\_\_\_\_\}$.
Then

$Log : D \longrightarrow \mathbb{C}$ is $\_\_\_\_\_\_$ and

$$\frac{d}{dz} Log\, z = \quad \ldots\ldots$$

Proof: Let $z = re^{i\theta} \in D$ where $\_\_\_\_\_\_\_\_\_$.
Then

$$Log\, z = \ldots\ldots\ldots$$

Let $u(r, \theta) = \qquad\qquad , \quad v(r, \theta) = \ldots\ldots$

So

$$\frac{\partial u}{\partial r} = \qquad\qquad \frac{\partial v}{\partial \theta} =$$

and $\boxed{\frac{\partial u}{\partial r} = \qquad}$.

$$\frac{\partial u}{\partial \theta} = \qquad\qquad , \quad \frac{\partial v}{\partial r} =$$

and $\boxed{\frac{\partial v}{\partial r} = \qquad}$

So the C-R eqns in polar form _ _ _ _ _ _ _ _ _.
and the partial derivatives

on the set of (r, θ) (_ _ _ _ region)

and we conclude that $Log\ z$ is _ _ _ _ _ _ on _ _ _)
and

$$\frac{d}{dz} Log\ z =$$

$$=$$

$\square$

<u>NOTE</u>: The function $Log: D \to \mathbb{C}$ is called
the _ _ _ _ _ _ _ _ _ _ _ _ of the
logarithm.

<u>Branches of the Logarithm Function</u>

Let $\alpha \in \mathbb{R}$. Let

$$D_\alpha := \{ \qquad \}$$

The Domain D_α

For $z \in D_\alpha$ define

$$\log_\alpha z :=$$

where .

NOTE $\log_\alpha z$ does NOT mean _ _ _ _ _ _ _ _ _ _ _.
This function is called a _ _ _ _ _ of the logarithm function.

Theorem: Let $\alpha \in \mathbb{R}$. The function

$$\log_\alpha : D_\alpha \longrightarrow \mathbb{C} \text{ is } \_\_\_\_$$

and

$$\frac{d}{dz} \log_\alpha(z) =$$

<u>Example</u> Let $\alpha = \pi/2$. Find

(i) $\log_{\pi/2}(-i)$ and (ii) $\log_{\pi/2}(1)$

<u>NOTE:</u> $\mathrm{Log}\, z =$.

<u>Properties</u> Let $z, z_1, z_2 \in \mathbb{C}$.

(1) $\exp(\log z) =$

(2) $\log(\exp(z)) =$

(3) $\mathrm{Log}(\exp(z)) =$

(4) $\log(z_1 z_2) =$

NOTE: In general it is _ _ _ _ that

$$\text{Log}\, z_1 z_2 = (\text{Log}\, z_1) + (\text{Log}\, z_2).$$

Example Let $z_1 = z_2 = -1 + i$.
Then

$|z_1| = |z_2| =$

$\text{Arg}\, z_1 = \text{Arg}\, z_2 =$

$z_1 z_2 =$

$|z_1 z_2| =$

$\text{Arg}\,(z_1 z_2) =$

$\text{Log}\,(z_1 z_2) =$

$\text{Log}\,(z_1) =$

$\text{Log}\,(z_2) =$

$\text{Log}\,(z_1) + \text{Log}\,(z_2) =$

Trigonometric Functions

Let $\theta \in \mathbb{R}$. Then

$$e^{i\theta} =$$

$$e^{-i\theta} =$$

$$2\cos\theta =$$

$$\cos\theta =$$

$$2i\sin\theta =$$

$$\sin\theta =$$

Definition Let $z \in \mathbb{C}$. We define

$\cos(z) :=$, and

$\sin(z) :=$.

Properties

(1) $\sin z$ & $\cos z$ are ______ functions.

(2) $\frac{d}{dz}\sin z =$, $\frac{d}{dz}\cos z =$.

(3) $\sin(z_1 + z_2) =$

(4) $\cos(z_1 + z_2) =$

(5) $\sin^2 z + \cos^2 z =$

(6) $\sin(z + \pi/2) =$

(7) $\sin(z + \pi) =$

(8) $\cos(z + \pi) =$

(9) $\sin(z + 2\pi) =$

$\cos(z + 2\pi) =$

(10) $|\sin z|^2 =$

(11) $|\cos z|^2 =$

Proof

(1) $\exp(z)$ is _ _ _ _ _ _ so $\exp(iz)$, $\exp(-iz)$ are _ _ _ _ _ _ functions. Hence

(2) $\sin z =$

$\frac{d}{dz} \sin z =$

Ex Show $\frac{d}{dz} \cos z = -\sin z$.

(3) $\sin(z_1 + z_2) =$

(5) $(\sin z)^2 + (\cos z)^2 =$

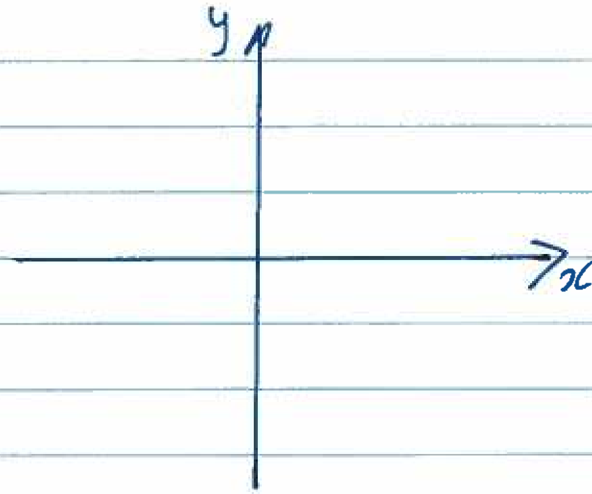
<u>Review of Hyperbolic Functions</u> Let $x \in \mathbb{R}$. Define

$\sinh x :=$, $\cosh x :=$

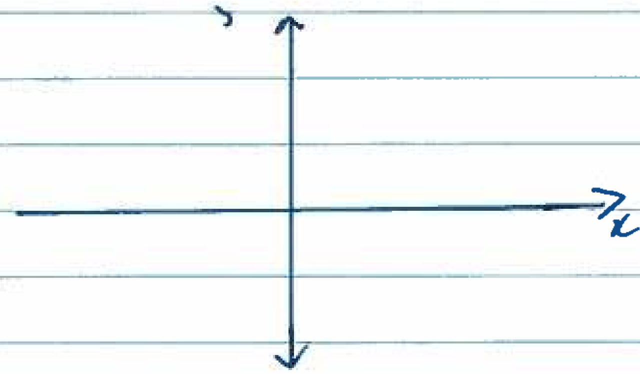
Then

$\frac{d}{dx} \sinh x =$, $\frac{d}{dx} \cosh x =$

$y = \sinh x$



$y = \cosh x$

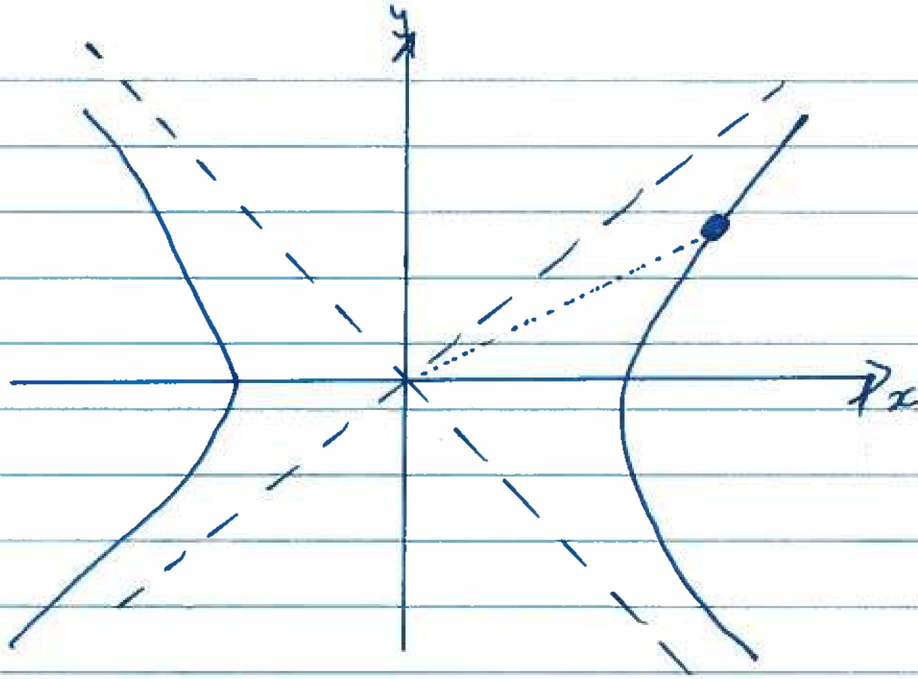


$\sinh^2 x =$

$=$

$\cosh^2 x =$

$\cosh^2 x - \sinh^2 x =$ (for $x \in$)



The _ _ _ _ $x^2 - y^2 = 1$.

<u>Note</u> For $t \in \mathbb{R}$ the point
$(\cosh t, \sinh t)$ lies on _ _ _ _ _ _ _ _.

<u>Lemma</u> Let $z = x + iy$ (where $x, y \in \mathbb{R}$). Then

$$\overline{\exp(z)} =$$

<u>Proof</u>:

$$\exp(z) =$$

$$\overline{\exp(z)} =$$

$\boxed{\text{Proof of (10): } \quad |\sin z|^2 = \sin^2 x + \sinh^2 y.}$

$$|\sin z|^2 =$$

NOTE: If $x \in \mathbb{R}$ then $|\sin x|$ _____.
HOWEVER it is _____ true that $|\sin z|$
for all $z \in \mathbb{C}$.

As $y \to \infty$ $\sin z \to$ ___ since
$\sinh y \to$ ___.

Proposition

(1) $\sin z = 0$ if & only if $z =$

(2) $\cos z = 0$ if & only if $z =$

PROOF (1) $|\sin z|^2 =$ where $z = x + iy$, $x, y \in \mathbb{R}$.

So $\sin z = 0$ iff

Example Solve $\sin z = 2$.

The Power Function

Definition Let $z \in \mathbb{C}$, $z \neq \ldots$, $c \in \mathbb{C}$. The power function z^c is defined by

$$z^c := \ldots\ldots\ldots\ldots$$

This is a .... function.

The principal value of z^c denoted by ....

is defined by

$$\ldots\ldots = \ldots\ldots\ldots\ldots$$

Let $\alpha \in \mathbb{R}$ & consider the branch of the logarithm

$$\log_\alpha z = \ldots\ldots\ldots\ldots$$

where ....

The corresponding branch of z^c is

$$z^c = \ldots\ldots\ldots\ldots$$

This function is analytic on $D_\alpha = \{ \ldots\ldots\ldots \}$

and $\frac{d}{dz} \log_\alpha z =$ for $z \in D_\alpha$.

Also

$$\frac{d}{dz} z^c =$$

Example Find i^i and P.V, i^i.