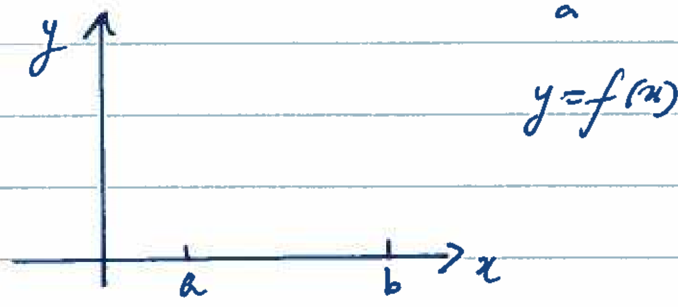
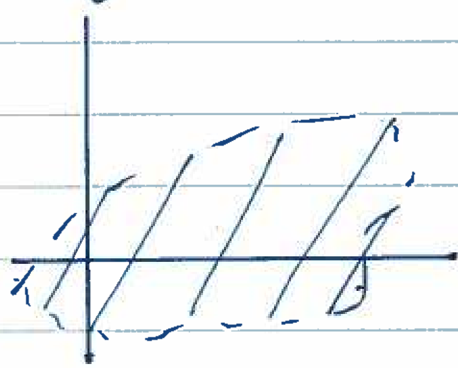
Chapter 4 INTEGRALS

Let $a, b \in \mathbb{R}$, $a < b$ and suppose $f : [a,b] \to \mathbb{R}$ is ________________. Then the Riemann integral $\int_a^b f(t)\,dt$ __________.



Suppose $f : D \to \mathbb{C}$ is a function where $[a,b] \subset D$



Let

$f(z) = u(z) + i\,v(z)$

so that

u:

v:

Assume u, v are __________.

We define the definite integral

$\int_a^b f(t)\,dt$ by

$$\int_a^b f(t)\,dt :=$$

Example Find $\int_0^{\pi/4} e^{it}\,dt$

Definition Let $w: (a,b) \longrightarrow \mathbb{C}$ and let $w(t) = u(t) + i\,v(t)$ so that

$u:$ $v:$

If u, v are _ _ _. we define

$w'(t) :=$

Example Let $w(t) = e^{it}$, for $t \in \mathbb{R}$.
Find $w'(t)$.

Proposition

Suppose $f : D \to \mathbb{C}$ is analytic and $(a, b) \subset D$.

Let $f(z) = u(z) + i\, v(z)$ for $z \in D$ &

$\omega(t) = u(t) + i v(t)$ for __ t _ _ . .

Then u, v are _ _ _ _ _ _ _ . _ _ and

$$f'(\ ) = \qquad .$$

PROOF (EX).

Proposition Let D be a domain, $f : D \to \mathbb{C}$ and

_ _ _ _ _ _ _ . . , Suppose f is _ _ _ _

_ _ _ _ _ _ _ _ _ _ _ . Then

(1) $\operatorname{Re}\left(\int_a^b f(t)\,dt\right) =$

(2) $\operatorname{Im}\left(\int_a^b f(t)\,dt\right) =$

(3) $\int_a^b \gamma f(t)\,dt =$

for _ _ _ _ _ _ _ .

(4) $\left| \int_a^b f(t)\,dt \right|$

Proof of (4)

Case 1 $f(t)$ is real-valued.

Case 2 $\int_a^b f(t)\,dt = 0.$

Case 3 $\int_a^b f(t)\,dt \neq 0.$

<u>Proposition</u> Suppose $g : D \to \mathbb{C}$ is analytic where D is a domain & _ _ _ _ _, and

$$= f(z)$$

for $z \in D$. Then

$$\int_a^b f(t)\,dt =$$

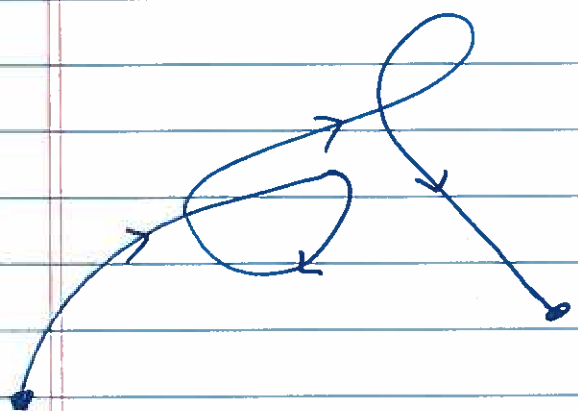
PROOF :

(p. 7)

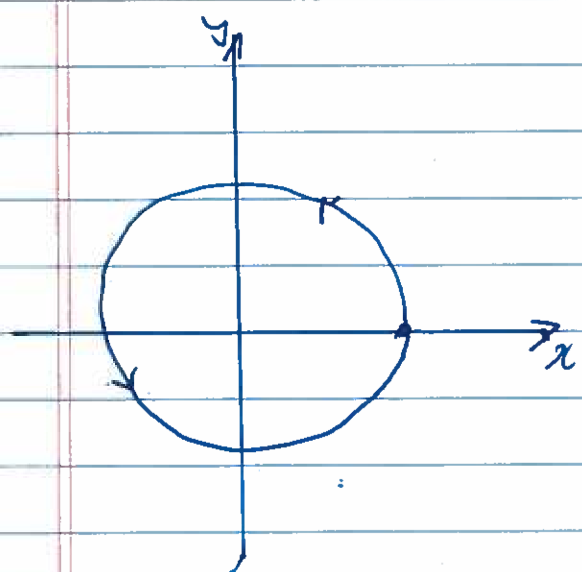
Example Compute $\displaystyle\int_0^{\pi/4} e^{-it}\,dt.$

Curves & Contours in $\mathbb{R}^2$ & $\mathbb{C}$

In general a curve in $\mathbb{R}^2$ is given

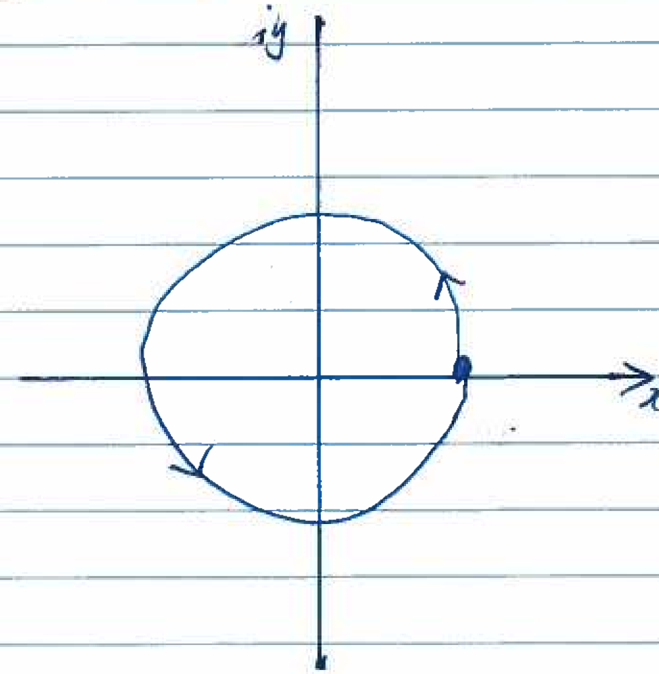


Example



A curve (or arc) in $\mathbb{C}$ is given

Example

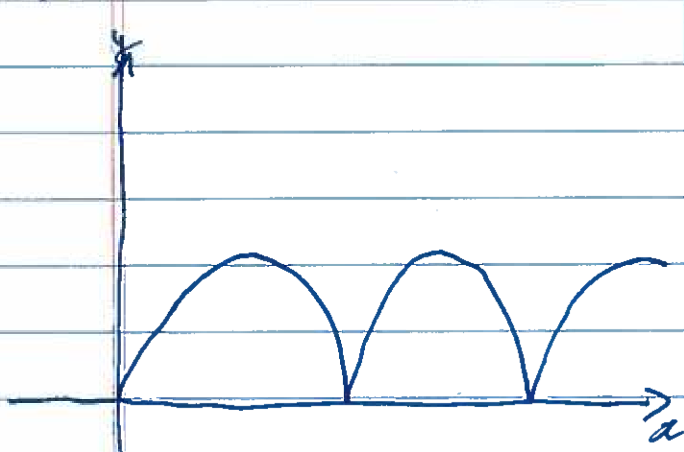


A curve is smooth if it has

Example The curve parametrized by

$x = t - \sin t$, $y = 1 - \cos t$ $(t \geq 0)$

is



$x' =$

$y' =$

$(x', y') =$

A curve in $\mathbb{C}$ is smooth if it has

Recall If

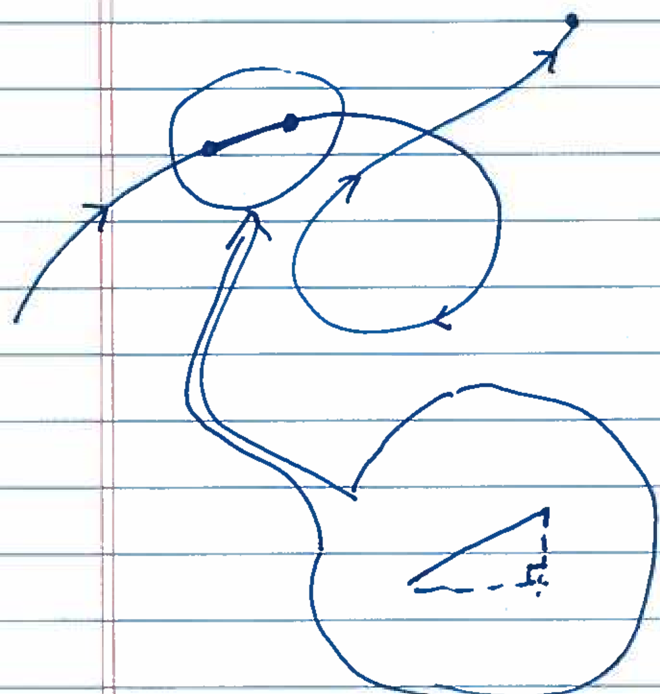
$z(t) = x(t) + iy(t)$,

then

$z'(t) =$

Arc Length

Suppose we are given a smooth curve $\mathcal{C}$ given parametrically by $(x(t), y(t))$ for $a \le t \le b$



$ds =$

The Length of $\mathcal{C}$ (from $t=a$ to $t=b$) is

$$\int_{\mathcal{C}} ds :=$$

Suppose $\mathcal{C}$ is a smooth curve parametrically by

$z(t) =$

for $a \le t \le b$.

$z'(t) :=$

$|z'(t)| =$

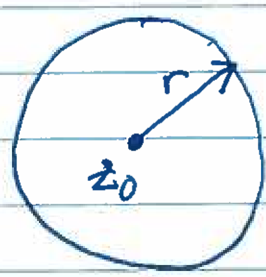
& the Length of $\mathcal{C}$ is

Parametrizations of Curves in $\mathbb{C}$

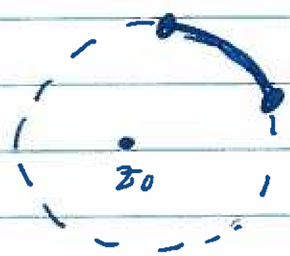
(1) Circles

The circle center z_0 and radius can be parametrized by

$z(t) =$



(2) Arc of a circle

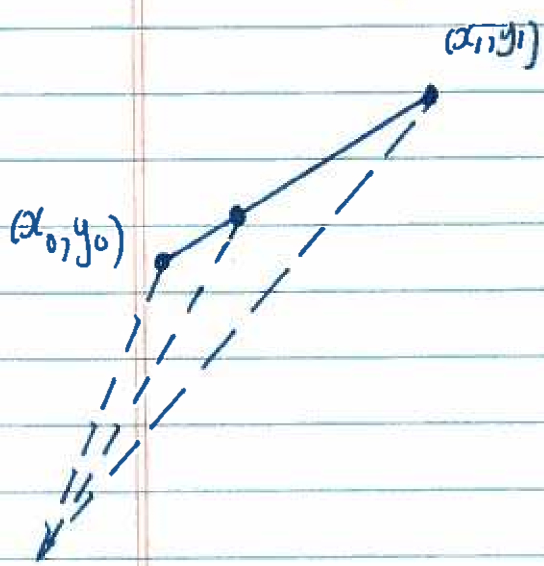


$z(t) =$

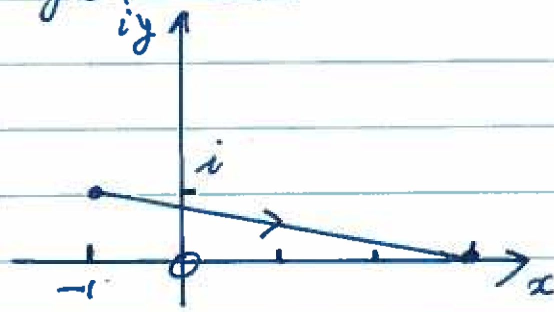
(3) Line segments

The line segment from $z_0 = x_0 + iy_0$ to $z_1 = x_1 + iy_1$ is parametrized by

$z(t) =$

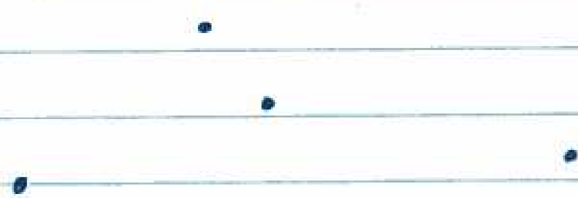


<u>Example</u> Find a parametrization of the line segment from $-1+i$ to 3.



$z(t) =$

<u>Definition</u> A <u>contour</u> (or _ _ _ _ _ _ _ _ _.) is a



A <u>simple closed contour</u>

Contour Integrals

Let $\mathcal{C}$ be a contour with parametrization $z(t)$, $a \le t \le b$. We define the <u>contour integral</u>

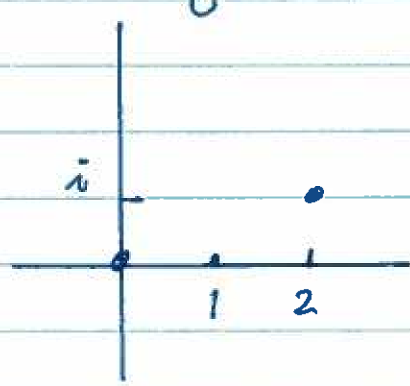
$$\int_{\mathcal{C}} f(z)\, dz :=$$

<u>NOTE</u>:

<u>Example</u> Let $\mathcal{C}$ be the line segment from 0 to $(2+i)$. Find the contour integral

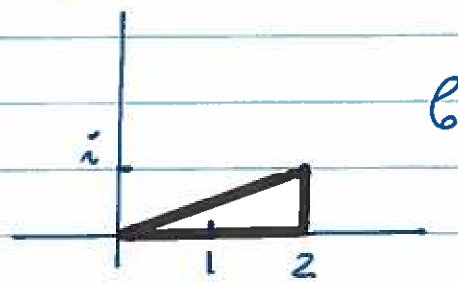
$$\int_{\mathcal{C}} z^2\, dz.$$

$\mathcal{C}$ is parametrized by



Hence $\displaystyle\int_{C} z^2\,dz =$

Example Let C be the simple closed contour given below.



Find $\displaystyle\int_{C} z^2\,dz.$

We already know that

$$\int_{C_1} z^2 \, dz =$$

C_2:

so

$$\int_{C_2} z^2 \, dz =$$

$\mathcal{C}_3$:

$$\int_{\mathcal{C}_3} z^2\,dz =$$

Hence $\displaystyle\int_{\mathcal{C}} z^2\,dz =$

<u>Properties</u>

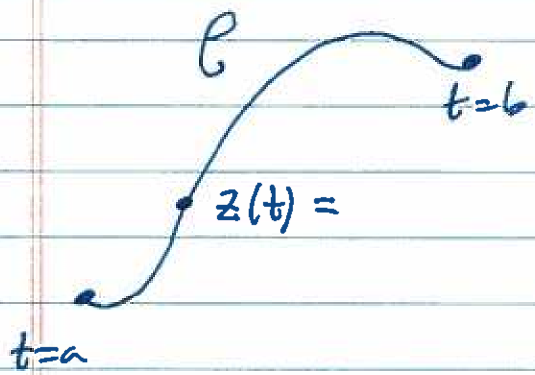
Let $\mathcal{C}$ be a contour with parametrization $\gamma : [a,b] \longrightarrow \mathbb{C}$. Then

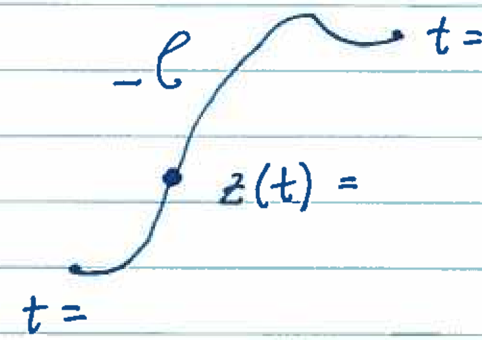
(1) $\displaystyle\int_{\mathcal{C}} \alpha f(z)\,dz =$

(2) $\displaystyle\int_{\mathcal{C}} [f(z) + g(z)]\,dz$

(3) $\displaystyle\int_{-\mathcal{C}} f(z)\,dz =$

~~Where~~ the $-C$ is parametrized by





(3) The length of the contour is

$$L =$$

assuming

(4) Suppose $|f(z)| \le M$ for all $z \in C$, and

L is the length of the contour C. Then

$$\left| \int_C f(z)\,dz \right| \le$$

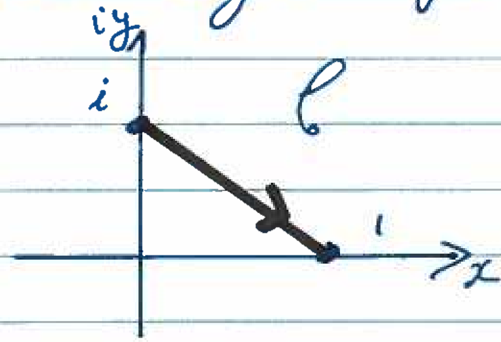
<u>NOTE</u>: In this Theorem we assume that

Proof of (4)

(p. 19)

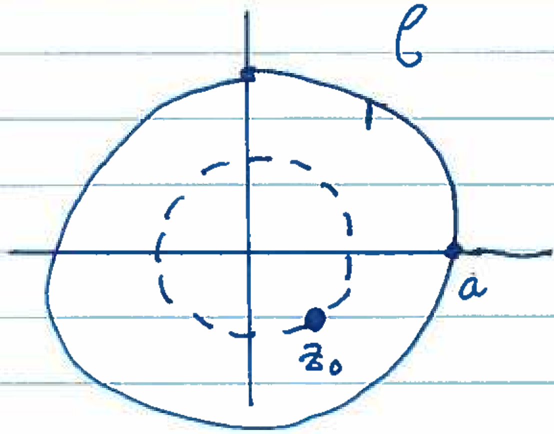
Example Find an upper bound for $\left| \int_{C} \frac{dz}{z^4} \right|$

where C is the line segment from i to 1.



<u>Example</u> Let $a > 0$, $z_0 \in \mathbb{C}$ and suppose
$0 < |z_0| < a$.
Let C be the simple closed circle $|z| = a$ with counterclockwise orientation. Show that

$$\left| \int_C \frac{dz}{z^2 + z_0^2} \right| \leq \frac{2\pi a}{a^2 - |z_0|^2}$$



<u>NOTE:</u> <u>when</u> $z^2 + z_0^2 = 0$

The Length of the Contour $= L =$
Suppose $z \in C$ so that

ANTIDERIVATIVES

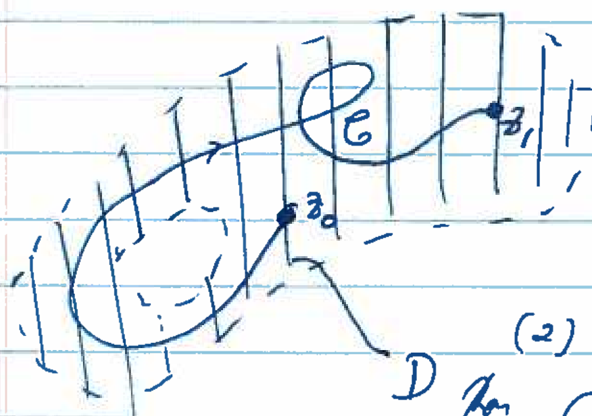
Suppose $f : D \to \mathbb{C}$ where $D \subset \mathbb{C}$ is a domain.
$F(z)$ is an anti-derivative of $f(z)$ if

____

____

Example An anti-derivative of $f(z) = z^2$
is

Proposition Suppose $D \subset \mathbb{C}$ is a domain, and
$F : D \to \mathbb{C}$ is ____________. Suppose the smooth
contour $\mathcal{C} \subset D$ is parametrized by $\gamma : [a,b] \to \mathbb{C}$
where $\gamma(a) =$ ____, $\gamma(b) =$ ____.
Then



(1) $\dfrac{d}{dt} F(\gamma(t))$

$=$ ______________

(2) If $F'(z) = f(z)$ for all $z \in D$,
then

$$\int_{\mathcal{C}} \_\_\_\_\_\_ \, dz = \_\_\_\_\_\_\_\_\_\_ .$$

Proof. Suppose (p.22)

$$F(z) = u(x,y) + iv(x,y), \quad \text{for } z = x+iy \in D.$$

Let $\gamma(t) = x(t) + iy(t)$ for $a \leq t \leq b$, & so

$$F(\gamma(t)) =$$

Then

$$\frac{d}{dt} u(x(t), y(t)) =$$

$$=$$

$$\frac{d}{dt} v(x(t), y(t)) =$$

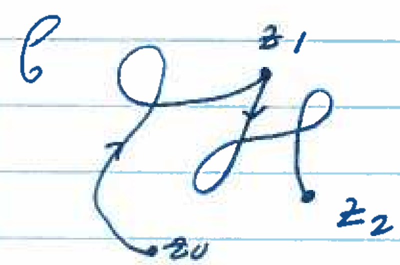
$$=$$

$$F'(z) =$$

$$F'(\gamma(t))\,\gamma'(t) =$$

(2) $\displaystyle\int_C f(z)\,dz =$

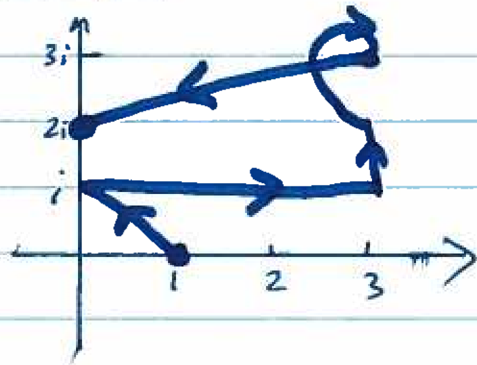
NOTE: (2) is also true for a general contour (piecewise smooth curve)



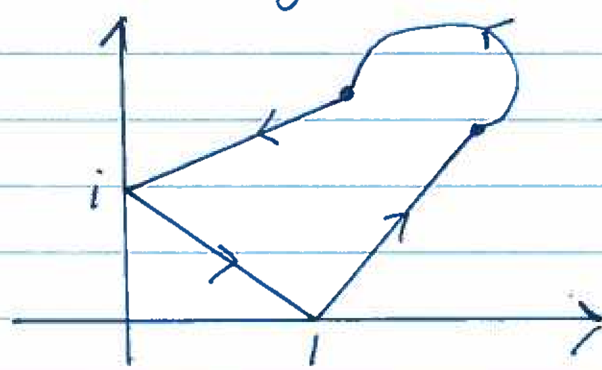
$\displaystyle\int_C f(z)\,dz =$

Example Let C be the contour below. Find

$$\int_C z^2 \, dz.$$



Closed Contours Find $\int_C \sin z \, dz$ when C is the closed contour given below.



<u>Proposition</u> Suppose $f(z)$ has an anti-derivative $F(z)$ on a domain D which contains the closed contour C.

Then

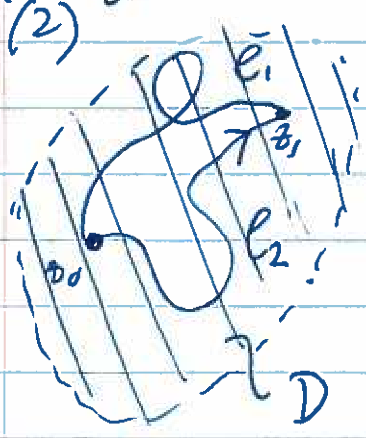
$$\int_C \text{-----}\, dz = \text{-----}.$$

<u>Proposition</u> Suppose $D \subset \mathbb{C}$ is a domain & $f: D \to \mathbb{C}$ is continuous.

The following statements are equivalent:

(1) f has an ---------------.

(2)

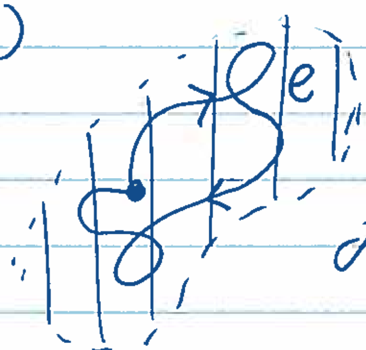


$$\int \qquad = \int$$

for ....

....

(3)



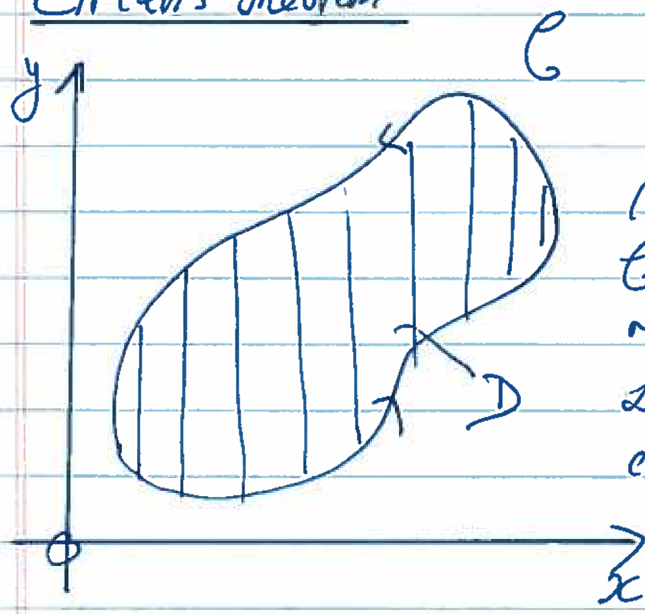
$$\int_C f(z)\,dz = 0$$

for ....

We have seen examples illustrating $(1) \Rightarrow (2)$ & $(1) \Rightarrow (3)$. These are easy to prove. For the remaining proofs see Section 49 of the Textbook.

CAUCHY'S INTEGRAL THEOREM

Green's Theorem



Let $\mathcal{C}$ be a simple closed contour in $\mathbb{R}^2$ with counter-clockwise orientation which is parametrizally $(x(t), y(t))$, $a \le t \le b$.
$\mathcal{C}$ is the boundary of a closed region D.
Let $P(x,y)$, $Q(x,y)$ be continuous real-value functions defined on D such that $\frac{\partial P}{\partial y}, \frac{\partial Q}{\partial x}$ are continuous on D.

Thm

$$\int_{\mathcal{C}} P\,dx + Q\,dy :=$$

$$=$$

Proof: Let $f : D \longrightarrow \mathbb{C}$ be analytic & suppose f' is continuous. Let

$$f(z) = u(x,y) + i\,v(x,y)$$

Then

$$\int_{C} f(z)\,dz =$$

$$\left[z = x(t) + i y(t), \quad dz = (\qquad\qquad) \, dt \right]$$

$$= \int_a^b (\qquad\qquad) dt + i \int_a^b (\qquad\qquad) dt$$

$$= \int_C (\qquad\qquad) + i \int_C (\qquad\qquad)$$

$$= \iint_D (\qquad\qquad) \, dx\,dy + i \iint_D (\qquad\qquad) \, dx\,dy$$

(by Green's Theorem since

....

....)

$$= \iint_D \Big(\qquad \Big)\, dx\,dy + i \iint_D \Big(\qquad \Big)\, dx\,dy$$

(by the ______________)

$$= \iint_D \qquad dx\,dy + i \iint_D \qquad dx\,dy$$

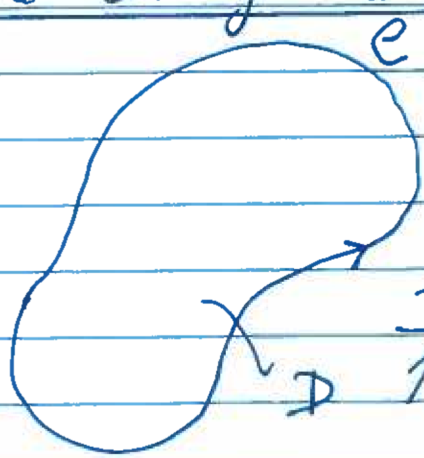
$=$

$=$

$\square$

Goursat (1858 – 1936) proved that the condition that f' be continuous is not necessary.

<u>The Cauchy – Goursat Theorem</u>



Let C be a ______ ______________ which is the ________ ______________ D.

If f is ________ on ____ then $\int_C f(z)\,dz =$ ____.

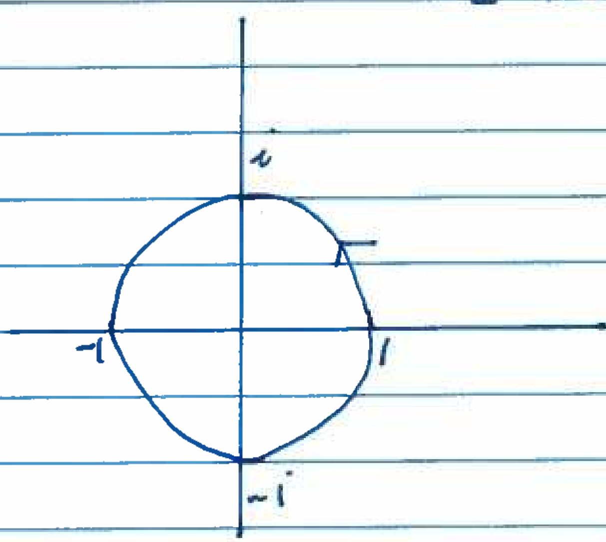
Example

Apply the Cauchy-Goursat Theorem to show that

$$\int_{C} f(z)\, dz = 0$$

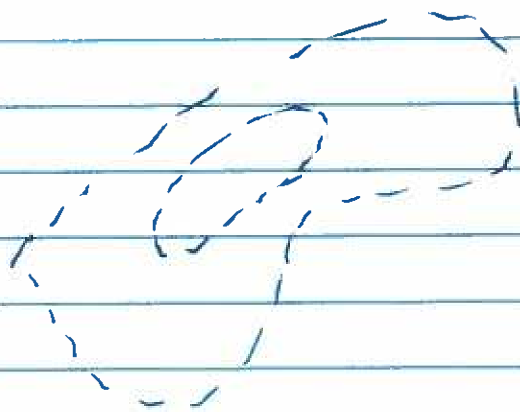
where C is the circle $|z|=1$ taken counter clockwise and

$$f(z) = \frac{1}{z^2+2z+2}$$



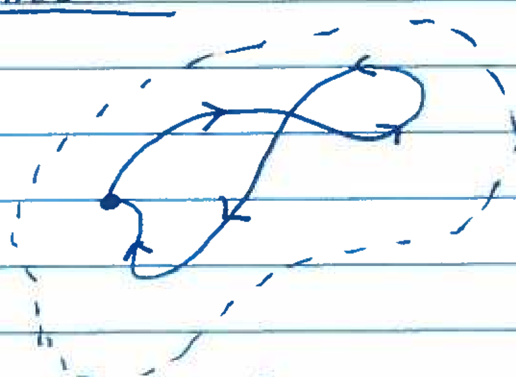
Multiply-Connected Domains

A domain D is <u>simply connected</u> if



A domain that is not simply connected is called ____________.

Theorem



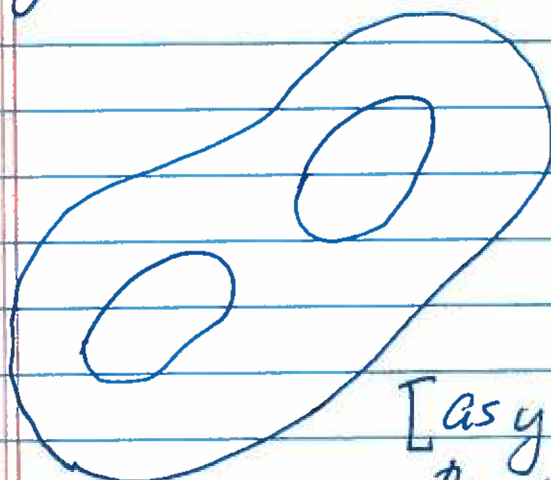
If f is analytic on a ____________ domain D, then

$$\int_C f(z)\,dz = 0$$

for ______________________.

Corollary Any analytic function on a ____________ domain has an ____________.

<u>Theorem</u> Let R be closed region whose boundary consists of ________ ________ contours C_j (each C_j ______) oriented so that points in R lie to the ______ of the boundary.
Let $\mathcal{B} = \cup C_j$ and suppose f is __________ on ______.

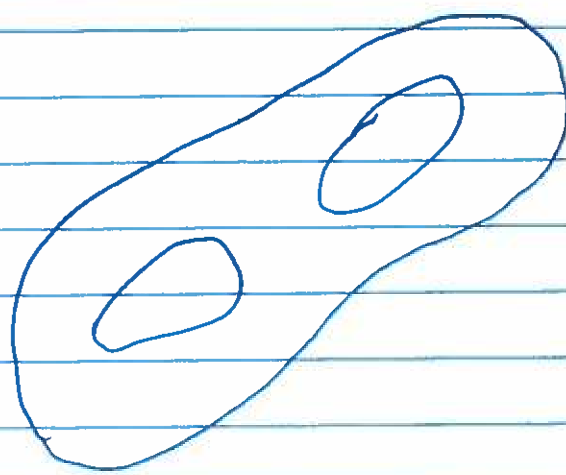


Then

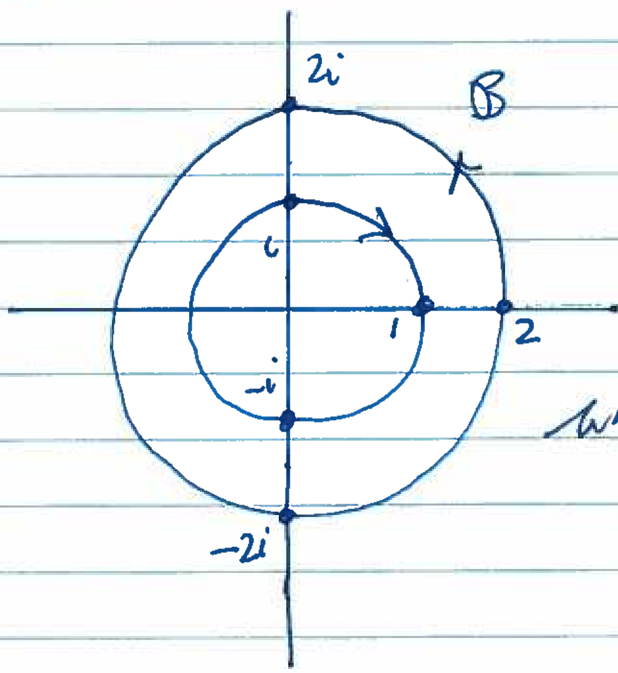
$$\int_{\mathcal{B}} f(z)\,dz = \underline{\qquad}.$$

[As you "walk" around the contour the region R is on your ________].

<u>PROOF FOR EXAMPLE</u>



Example Let $\mathcal{B}$ be the contour below.

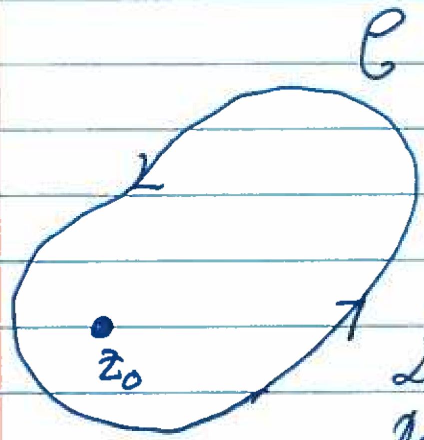


Show

$$\int_{\mathcal{B}} f(z)\,dz = 0$$

where $f(z) = \dfrac{1}{z^2(z^2+9)}$

The Cauchy Integral Formula



Suppose $f(z)$ is ______ everywhere ______ and ______ C (with ______ orientation).

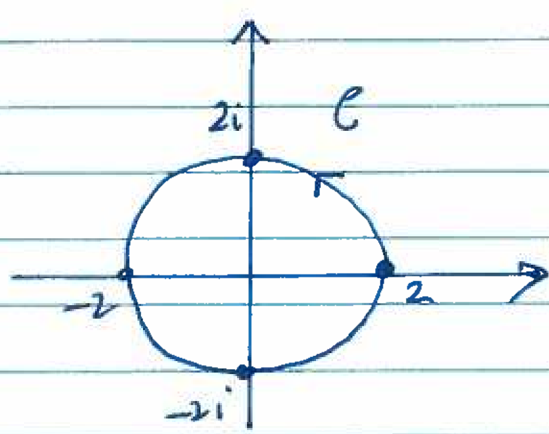
Let z_0 be any point ______ to the contour C.

Then

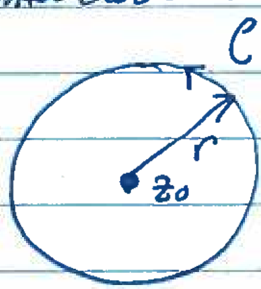
$$f(z_0) =$$

Example Let C be the circle $|z|=2$ (with counterclockwise orientation). Find

$$\int_C \frac{z\,dz}{(9-z^2)(z+i)}$$



<u>Lemma</u> Let C be the circle $|z - z_0| = r$ with counterclockwise orientation. Then



$$\int_C \frac{1}{z - z_0}\, dz =$$

<u>Proof:</u> C is parametrized by

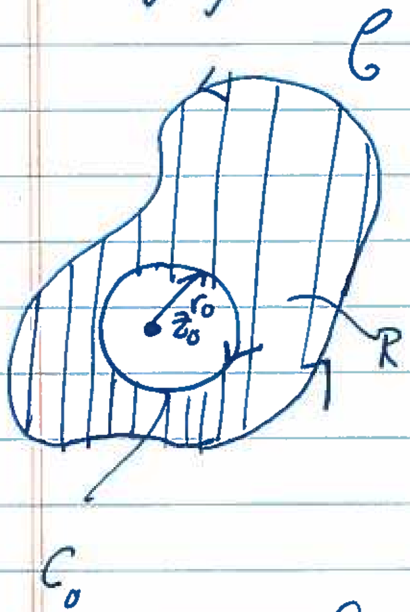
$z = \gamma(t) =$

Then $dz =$

So

$$\int_C \frac{dz}{z - z_0} =$$

Proof of the Cauchy Integral Formula



Choose r_0 small enough so the disc $D(z_0, r_0)$ is contained in the interior of C.

Let $\mathcal{B} = C \cup C_0$.

This contour bounds the region R and the function

$$\frac{f(z)}{z - z_0} \quad \text{is} \quad \_\_\_\_\_\_\_\_\_\_ .$$

Hence
$$\int_{\mathcal{B}} \frac{f(z)\,dz}{z - z_0} = \_\_\_\_\_ .$$

But

$$0 = \int_{\mathcal{B}} \frac{f(z)\,dz}{z - z_0} =$$

and so
$$\int_{C} \frac{f(z)}{z - z_0}\,dz =$$

$$=$$

$$= \qquad\qquad =$$

$=$

$=$

Now let $M_r = \max_{|z-z_0| \leq r} |f(z) - f(z_0)|$.

Since $f(z)$ is continuous at z_0 it can be shown that

$$\lim_{r \to 0} M_r =$$

Now,

$$\left| \int_C \frac{f(z)}{z-z_0} dz - 2\pi i f(z_0) \right|$$

$$= \left| \int_{-C} \frac{f(z)}{z-z_0} - f(z_0) \int_{-C} \right|$$

$=$

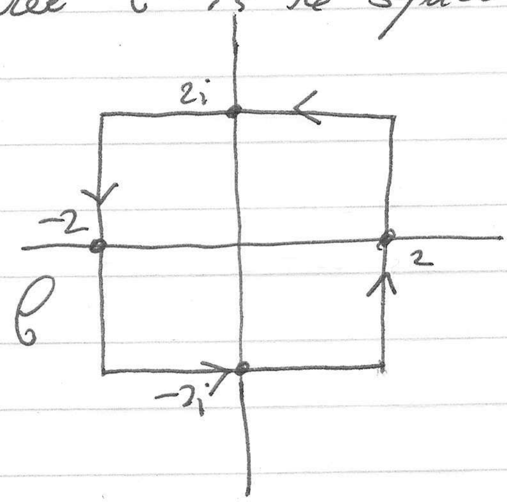
$\leq$

$=$

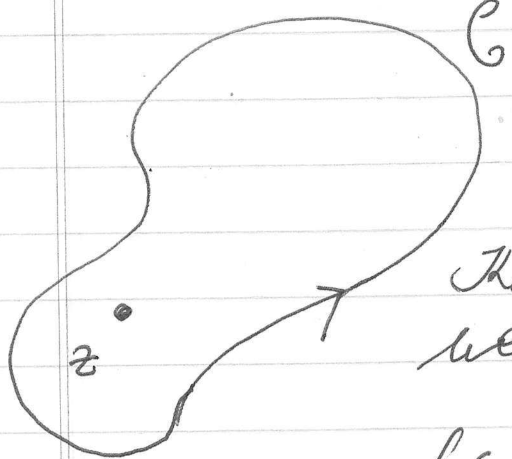
Letting $r \to 0$ we find that

Example Find $\displaystyle\int_C \frac{\cos z}{z(z^2+8)}\,dz$

where C is the square below.



Derivatives of Analytic Functions



Let f be ____ on a ________ contour and on its _____. Then by the Cauchy Integral Formula we have

$f(z) =$

and

$f'(z)$

NOTE: This can be proved rigorously. See ______. In a similar it can be shown that ____ is ______ on the interior of C and

$f''(z) =$

We have the following

Theorem Suppose f is ____ at the point $z_0 = x_0 + iy_0$ and suppose

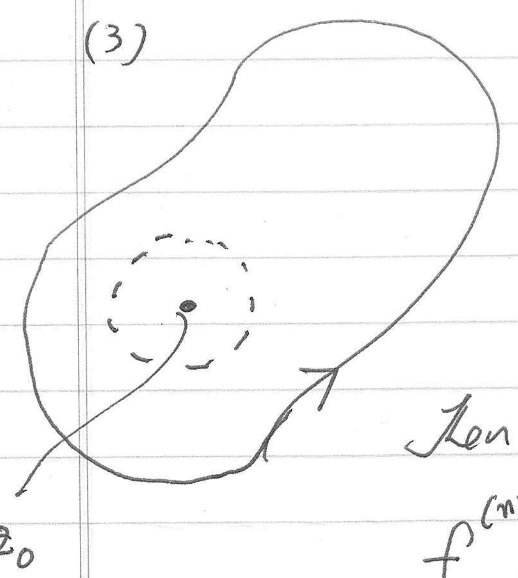
$$f(z) = u(x,y) + i\,v(x,y).$$

Then

(1) There is a ______ of z_0 for which the ______ of $f(z)$ to ________ ___ and are ________.

(2) The functions $u(x,y)$, $v(x,y)$ have ______ ________________________.

(3)



z_0

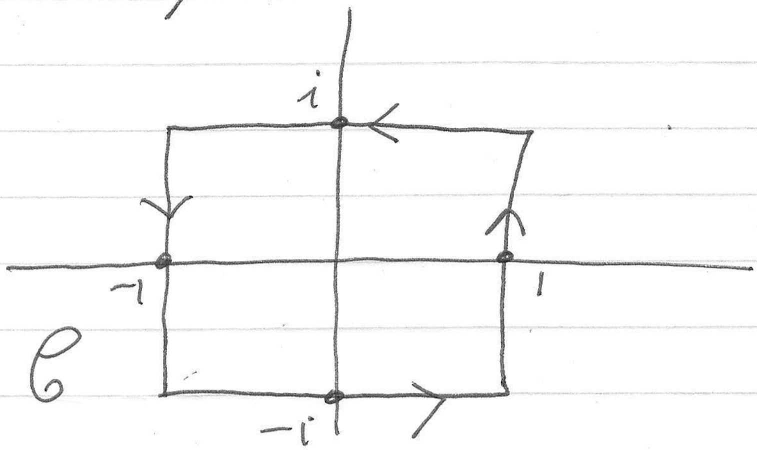
C Let C be a ______ ____ with ______ ______. Suppose $f(z)$ is ______ ______ ________ and suppose z_0 is ______.

Then

$$f^{(n)}(z_0) =$$

Example Find $\displaystyle\int_C \frac{\sinh z}{z^4}\, dz$ where

C is the square below.



(p.41)

Morera's Theorem [Morera (1856–1909)]

Let f be ______ on a domain D

and suppose

$$\int_C f(z)\,dz$$

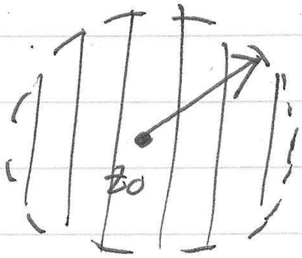
for ____________.

Then f is ______ ______.

PROOF:

MAXIMUM ABSOLUTE VALUE of FUNCTIONS

Lemma

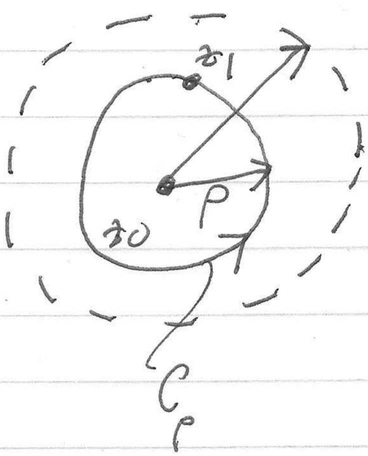


Suppose f is analytic on an open neighborhood $D(z_0, r)$ of z_0 & suppose

$$|f(z)| \leq |f(z_0)|$$

for all $z \in D(z_0, r)$. Then

PROOF:



Let $z_1 \in D(z_0, r)$, $z_1 \neq z_0$ so

$\rho = |z_1 - z_0|$ ...

Let C_ρ be the +ve oriented circle $|z - z_0| = \rho$.

The Maximum Modulus Principle

Suppose f is analytic on a domain D and suppose f is NOT a _ _ _ _ _ _.
Then

$|f(z)|$ has _ _ _ _ _ _ _ _;

ie There is no $z_0 \in D$ such that

Sketch of Proof

Suppose f is analytic on a domain D but it attains its maximum value on D, i.e. there is a $z_0 \in D$ such that

$$\_\_\_\_\_\_\_\_\_\_$$

________.

Let $A = \{z : z \in D \ \& \ f(z) = f(z_0)\} = f^{-1}(\{w_0\})$

where $w_0 =$ ______.

A is ________ since f is ________.

Let $z_1 \in D$. Choose $\delta > 0$ such that the disk $D(z_1, \delta)$ ______. Then

$$|f(z)| \ \_\_\_\_\_\_$$

for ______. The Lemma implies

$$\_\_\_\_\_\_\_\_\_\_$$

for ______ and

$$D(z_1, \delta) \ \_\_\_\_\_\_.$$

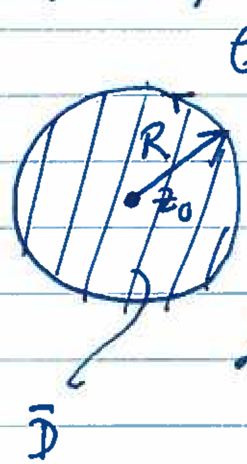
Hence A is ______.

So A is both ______ and ______ subset of D. By a theorem from Advanced Calc. this implies $A =$ ____

and hence f is ________.

<u>Corollary</u> Suppose $D \subset \mathbb{C}$ is a domain & $R = \overline{D}$ is bounded. Suppose f is analytic on D but ______ and f is ______ on R.
Then

<u>Cauchy's Inequality</u> Let $R > 0$ and suppose f is ______ on the closed disc



$\overline{D}(z_0, r) = \{ z :$ $\}$.

Let $\mathcal{C}$ be the boundary, i.e.

$\mathcal{C} = \{$ $\}$.

Let

$$M_R = \max_{z \in \mathcal{C}} |f(z)|$$

Then $|f^{(n)}(z_0)| \leq$.

PROOF:

Liouville's Theorem If f is __ __ __ __
and __ __ __ __ __ then __ __ __ __ __ __ __.

PROOF:

<u>The Fundamental Theorem of Algebra</u>

Let $P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$

be a complex polynomial of degree $n \geq 1$.

Here a_j ___ and a_n ___. Then

the equation

___

has ___.

PROOF:

(p. 49).

p. 50)

<u>Corollary</u> Every complex polynomial $P(z)$ of degree $n \geq 1$ can be _ _ _ _

$P(z) =$

for some