Chapter 5 – Series.

Defn: Convergence of a complex sequence.

An infinite sequence

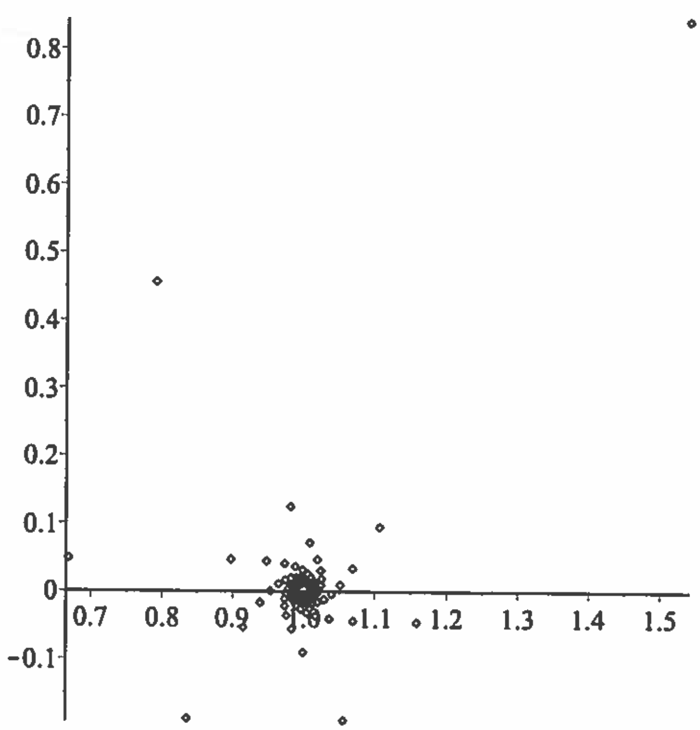
$$z_1, z_2, \dots$$

of complex numbers also written as ___ converges to a complex number w if

$$\lim_{n\to\infty}$$

We say the sequence ______ has a ______, and write

Example Let $z_n = 1 + \frac{e^{in}}{n}$ for $n = 1, 2, \dots$



Def. Convergence of a Complex Series.

We say a complex series

$$z_1 + z_2 + z_3 + \cdots\cdots \quad =$$

converges to a complex number w if the sequence of ________

$s_1 =$, $s_2 =$, $s_3 =$, $\ldots$

ie

$$s_n =$$

converges to ____.

In this case we write

Example If ______ then the geometric series

$$\sum_{n=0}^{\infty}$$

__________ and

$$\sum_{n=0}^{\infty} \quad =$$

Power Series Let z_0 be a complex constant. A power series in $(z - z_0)$ is a series of the form

Example The series

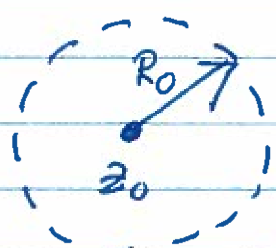
$$\sum_{n=0}^{\infty} \frac{z^n}{n!} =$$

______ for ______ and

$$\sum_{n=0}^{\infty} \frac{z^n}{n!} =$$

TAYLOR'S THEOREM

Let $z_0 \in \mathbb{C}$, $R_0 > 0$. Suppose $f(z)$ is ______ on the open disc $D(z_0, R_0)$. Then the power series



$$\sum_{n=0}^{\infty}$$

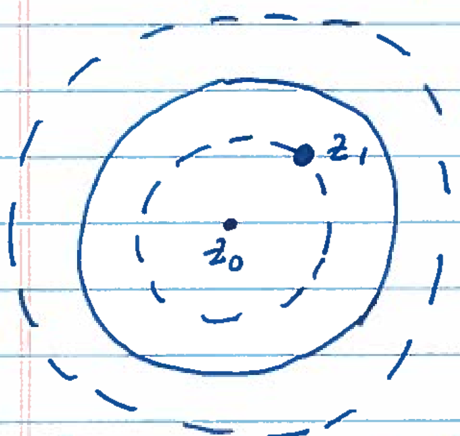
______ for ______, and

$$\sum_{n=0}^{\infty} \qquad =$$

for

<u>Example</u> $f(z) = e^z$.

Sketch of the Proof of Taylor's Theorem



Let $z_1 \in D(z_0, R_0)$ be fixed.

Choose

$|z_0 - z_1| < R < R_0$

Let C be the simple closed circle

$|z - z_0| = \_\_\_$.

By Cauchy Integral Formula

$$f(z_1) =$$

It can be shown that

$$\frac{1}{z - z_1} = \sum_{j=0}^{n-1} \frac{(z_1 - z_0)^j}{(z - z_0)^{j+1}} + \frac{1}{(z - z_1)} \left(\frac{z_1 - z_0}{z - z_0}\right)^n.$$

Then

$$f(z_1) = \frac{1}{2\pi i}\int_C$$

$$= \sum_{j=0}^{n-1} \frac{1}{2\pi i}\int_C$$

$$+ \left(\frac{1}{2\pi i}\int_C \qquad dz\right)(z_1 - z_0)^n$$

$$= \sum_{j=0}^{n-1}$$

$$+ \quad R_n(z_1)$$

where

$$R_n(z_1) = \frac{(z_1 - z_0)^n}{2\pi i}\int_C \qquad dz$$

Suppose $\quad |f(z)| \leqslant M \;$ for $\; z \in C.$

So

$$\left|\frac{f(z)}{\qquad}\right| = \frac{|f(z)|}{\qquad} \leqslant$$

Hence

$$|R_n(z_1)| = \left| \frac{(z_1 - z_0)^n}{2\pi i} \int_C \qquad dz \right|$$

$$\leq$$

$$=$$

Now $0 \leq R_1 < R$ so that $0 \leq \frac{R_1}{R} < 1$

and

$$\lim_{n \to \infty} \left(\frac{R_1}{R}\right)^n =$$

and

$$\lim_{n \to \infty} R_n(z_1) = 0$$

and therefore

$$f(z_1) =$$

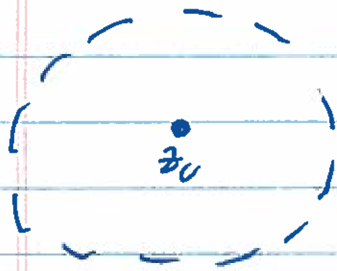
and

$$f(z) =$$

for all $z \in$ ____ $\square$

Theorem (Uniqueness of Taylor Series)

Suppose f is analytic for $|z - z_0| < r$ where $r > 0$. Then for ___ .



$$f(z) =$$

Conversely if

$$f(z) = \sum_{n=0}^{\infty} a_n \, (z - z_0)^n$$

for ____ then

$$a_n =$$

for $n = 0, 1, 2, \ldots\ldots$

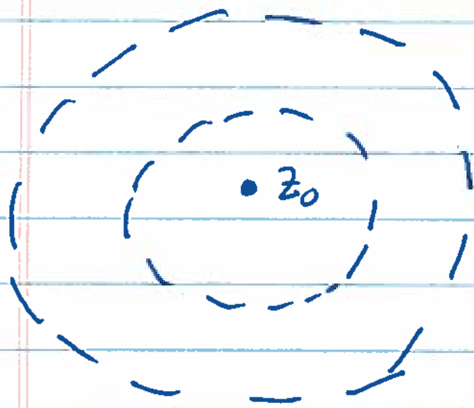
Examples Find the Maclaurin Series (ie $z_0 =$ ___) and a domain of convergence

(1) $\dfrac{1}{1-z}$

(2) $\dfrac{1}{1+z}$

(3) $\dfrac{1}{1+z^2}$

Laurent's Theorem (1842).



Suppose $f(z)$ is _ _ _ _
on the annulus

$$r_2 < |z - z_0| < r_1$$

Let C be _ _ _ _ _ _ _ _.
_ _ _ _ _ _ _ _ _ _ _ .

Then f has an expansion

$f(z) =$ for

(*) $=$

where

$a_n =$

$b_n =$

The expansion in (*) is called a _ _ _ _ _ series
and it is _ _ _ _ _ _ _.

Example Find the Laurent series about $z=0$ for

$$f(z) = \frac{1}{z(z-1)}$$

(i) Valid for $0 < |z| < 1$.
(ii) Valid for $1 < |z| < \infty$.