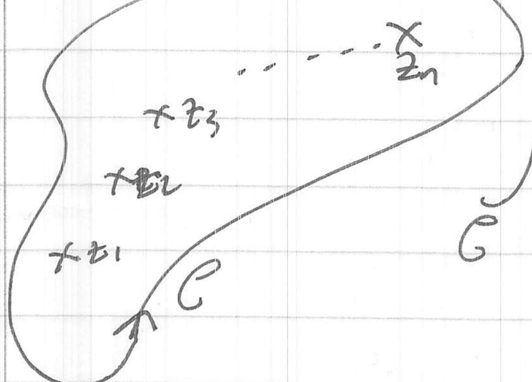
Residue at Infinity

Theorem: Let C be a simple closed contour with positive orientation. Suppose $f(z)$ is _ _ _ _ _ _ _ _ _ except for finitely many singularities $z_1, z_2, \ldots, z_n$ _ _ _ _ _ _ _ .

Then

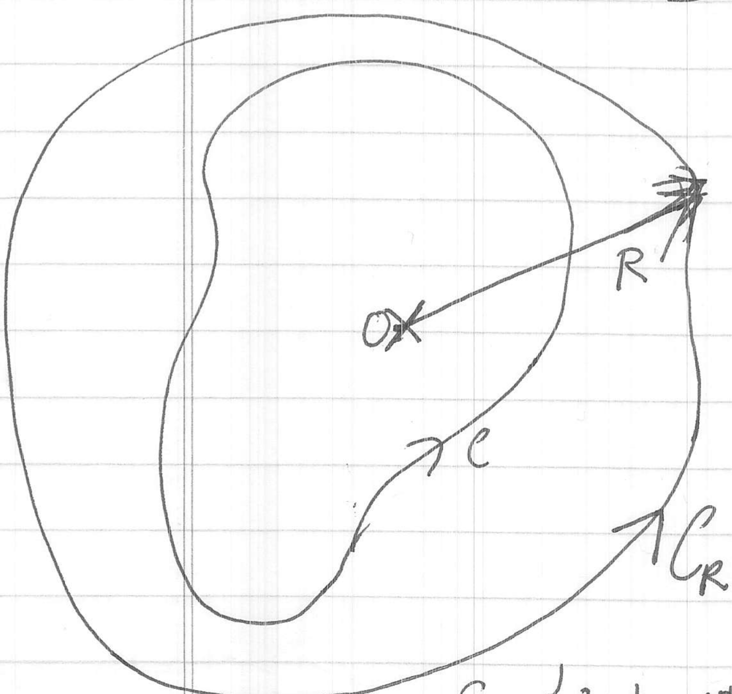


$$\int_C f(z)\,dz =$$

PROOF: Choose $R > 0$ such that

$C \subset D(0, R)$. Then $f(z)$ is analytic on the

_ _ _ _ _ _ _ _ _ _ _ _ _ _ . . .



Hence

$$\int_C f(z)\,dz =$$

(by a Corollary to

$C_R : \{ z : |z| = R \}$

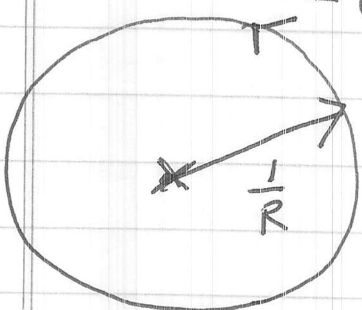
(17)

Let C_R : $z = Re^{it}$, $0 \le t \le 2\pi$.
Then $dz =$

$$\int_{C_R} f(z)\,dz =$$

Let $g(z) = \frac{1}{z^2} f\left(\frac{1}{z}\right)$.

Let $\tilde{C}_{R'}$: $z = \frac{1}{R} e^{-it}$, $0 \le t \le 2\pi$.



Then

$$\int_{-\tilde{C}_{R'}} g(z)\,dz = -\int_{\tilde{C}_{R'}} g(z)\,dz$$

$=$

$g(z) = \frac{1}{z^2} f\left(\frac{1}{z}\right)$ is analytic for $0 < |z| \leq \frac{1}{R}$

since when $0 < |z| \leq \frac{1}{R}$, $\left|\frac{1}{z}\right| \geqslant$

and $f\left(\frac{1}{z}\right)$ is analytic for ---.

The only singularity of $g(z)$ inside $-\tilde{C}_{R'}$ is

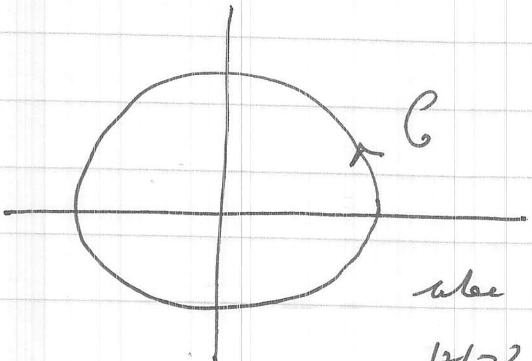
at $z =$ ---. Hence

$$\int_{-\tilde{C}_{R'}} g(z)\, dz =$$

and therefore

$$\int_{C} f(z)\, dz =$$

Example (#3, p.247) Find



$$\int_C \frac{4z-5}{z(z-1)}\, dz$$

where C is the simple closed circle $|z|=2$ with +ve orientation

using (a) Cauchy Residue Theorem,
(b) Residue at infinity

Definition Suppose f is analytic at z_0 and z_0 is a zero of f ie $f(z_0) = 0$. We say z_0 is a **zero of order m** if

This means f has a Taylor expansion near $z = z_0$

$f(z) =$

$=$

$=$

where $g(z) =$

is _ _ _ _ _ _ and $g(z_0) =$

Theorem Let f be analytic at z_0. The following are equivalent

(i) f has a zero of order m at z_0.

(ii) There is a function g which is

<u>Example</u> The function $f(z) = \sin z$ has zeros at

$z =$

$f'(n\pi) =$

Hence all the zeros of f have _____.

We say these zeros are _____. For example,

$\sin z =$

$=$

and $z = 0$ is a _____.

<u>Example</u> $f(z) = 1 - \cos z$

$=$

$=$

$=$

and $z =$ __ is a _____.

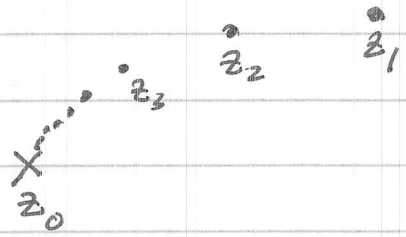
NOTATION. Let $f : D \to \mathbb{C}$. We write $f \equiv 0$ and say _____ if

<u>THEOREM</u>: Let $D \subset \mathbb{C}$ be a domain (i.e. _____)

Suppose $f : D \to \mathbb{C}$ is analytic. Let

$$Z = Z(f) = \{ z \in D : \_\_\_ \}.$$

If there is a $z_0 \in Z$ and a sequence $\{z_n\}_{n=1}^{\infty} \subset$ ____ such that _____. Then $f \equiv 0$.



PROOF: Claim 1: $f^{(n)}(z_0) = 0$ for all $n = 0, 1, 2, \ldots$

Proof of Claim: Suppose by way of contradiction that

$$f(z) = \sum_{n=n_0}^{\infty}$$

for $|z - z_0| < \delta$, some $\delta > 0$.

Let

$$g(z) = \sum_{n=n_0}^{\infty}$$

$$=$$

for $|z - z_0| < \delta$ and

$$g(z_0) =$$

$f(z) =$ ___ ___ ___ ___ . for $|z - z_0| < \delta$

$z_n \in$ ___ ___ ___ and $z_n \neq$ ___ ___ ___ for $n \geq 1$

so that

___ $= f(z_n) =$ ___ ___ ___ ___ and $g(z_n) =$ for ___

But $g(z)$ is analytic & hence ___ ___ ___ for $|z - z_0| < \delta$

so $\lim_{n \to \infty} g(z_n) =$ and

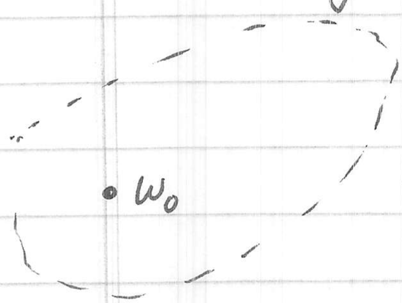
$$g(z_0) =$$

which is a ___ ___ ___. So Claim 1 ___ ___ ___.

Hence $f^{(n)}(z_0) =$ for
and the set
$$E = \{ z \in D : f^{(n)}(z) = 0 \text{ for } \_\_\_ \}$$
is ____ (since ____).
Let
$$E_n = \{ z \in D : f^{(n)}(z) = 0 \}.$$
Then each E_n is ____ since each $f^{(n)}$ is ____ and
$$E = \qquad \text{is a } \_\_\_ \text{ subset of } D.$$

<u>Claim 2</u> E is an open subset of D.

<u>Proof of Claim</u>: Let $w_0 \in E$. Then $w_0 \in D$ and $f^{(n)}(w_0) =$



There is an $\varepsilon > 0$ such that
$$f(z) =$$
for all $|z - w_0| < \varepsilon$. Hence ____ and E is ____.

Since E is an nonempty, closed and open subset of D and D is ____, it follows that $E =$ ____; ie $f(z) =$

and

$\square$

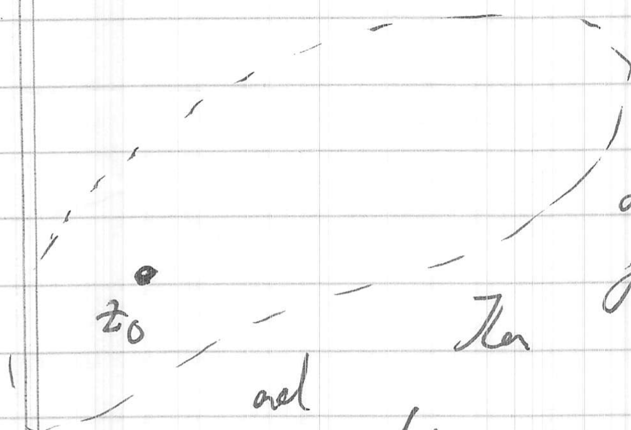
Corollary Let $D \subset \mathbb{C}$ be a domain.
Suppose $f : D \longrightarrow \mathbb{C}$ is analytic,
$D(z_0, r) \subset D$ and

$$f(z) = 0 \quad \text{for} \ \_\_\_\_\_.$$

Then

$$f \equiv \_\_\_.$$

PROOF.



$D(z_0, r) \subset$ ___.

Let $z_n =$ ____

for .

Then $\{z_n\} \subset$ ____ . . . ,

and

$\lim_{n\to\infty} z_n =$. The Theorem implies $f \equiv 0$. □

Corollary Let $D \subset \mathbb{C}$ be a domain.
Suppose $f : D \longrightarrow \mathbb{C}$ is ____ and
$\mathbb{R} \subset D$ and

$$f(z) = 0 \quad \text{for} \ \_\_\_\_.$$

Then

PROOF: $\mathbb{R} \subset$ ____.

Let $z_n =$ for $n \geq 1$.

Then $\{z_n\} \subset$ ____

$\lim_{n\to\infty} z_n =$

The Theorem implies ____. □

Example PROVE $\sin^2 z + \cos^2 z = 1$ for all $z \in \mathbb{C}$.

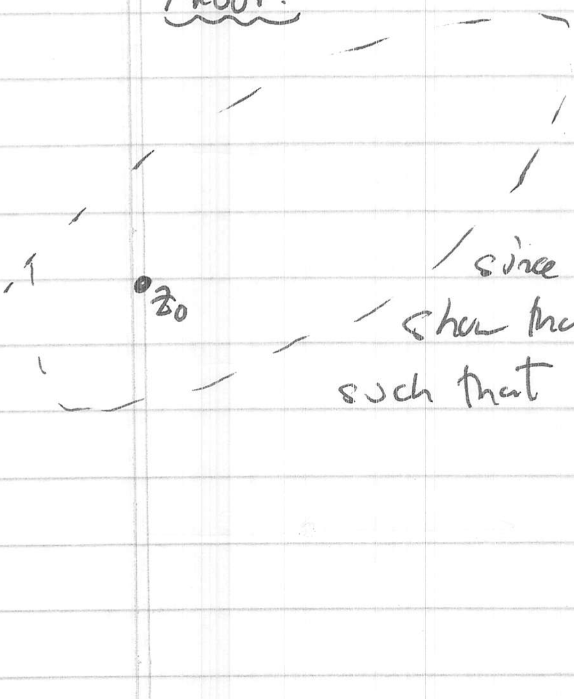
PROOF:

Corollary Let $D \subset \mathbb{C}$ be a domain. Suppose $f : D \to \mathbb{C}$ is _____ and $f \not\equiv 0$. Then ____ zero of f is _______.

PROOF: Suppose hypotheses hold. Suppose z_0 is a zero of f. Then

$$D(z_0, \delta) \subset D \text{ for some } \delta > 0,$$

since D is _____. We want to show that there is a $0 < \delta < \delta_0$ such that



Suppose by way of contradiction that for each $0 < 1/n < \delta$

there is

Then

and

$$\lim_{n \to \infty} z_n$$

and $\{z_n\} \subset$

The theorem implies