Chapter 6 Residues & Poles

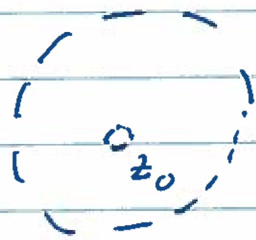
Definition: The point z_0 is called singular point of $f(z)$ if ______________________________

Examples

① $f(z) = \dfrac{\cos z}{z(z+1)}$

② $f(z) = \mathrm{Log}(z+3)$

<u>Definition</u> A singular point z_0 of $f(z)$ is <u>isolated</u> if _



<u>Examples</u>

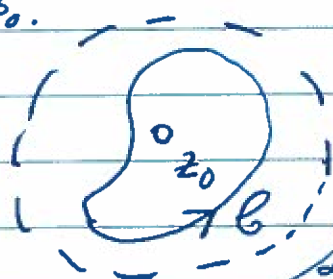
① $f(z) = \dfrac{\cos z}{z(z+1)}$

② $f(z) = \operatorname{Log}(z+3)$.

③ $f(z) = \dfrac{1}{\sin(\pi/z)}$

Using Laurent Series to Compute a Contour Integral

Suppose $f(z)$ has an isolated singularity at z_0. Then for some ___



$f(z)$ is ____

for ______.

So $f(z)$ has a Laurent Expansion.

$f(z) =$

and

$b_n =$

Thus

$b_1 =$

and $\int_{C} f(z)\, dz =$

The coefficient b_1 is given a special name.

If z_0 is an ____ singularity of $f(z)$ then the residue of f at z_0 is ____ of ________ in the ______

of ____ near ____.

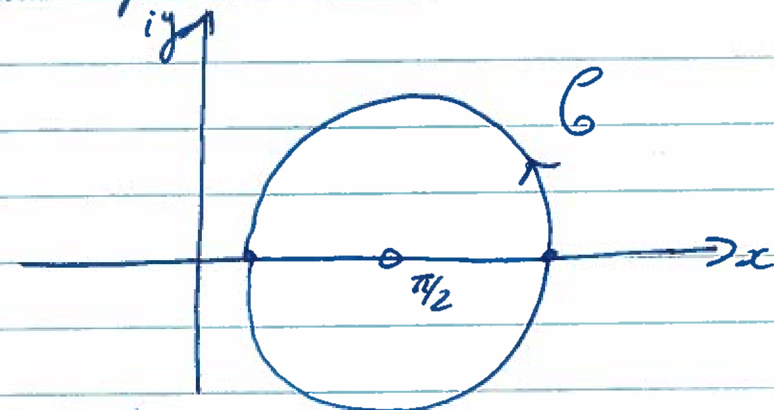
We write

for the residue of $f(z)$ at $z=z_0$.

<u>Example</u> Use Residues to compute

$$\int_C \frac{\cos z}{(2z-\pi)^2}\,dz$$

where C is the simple closed circle $|z-\pi/2|=1$ with positive orientation.



The function $f(z)=\dfrac{\cos z}{(2z-\pi)^2}$ is analytic

everywhere except at ___ which

______. so

$$\int_C f(z)\,dz =$$

We need to find the coefficient of ____ in the Laurent expansion of ____ near ____.

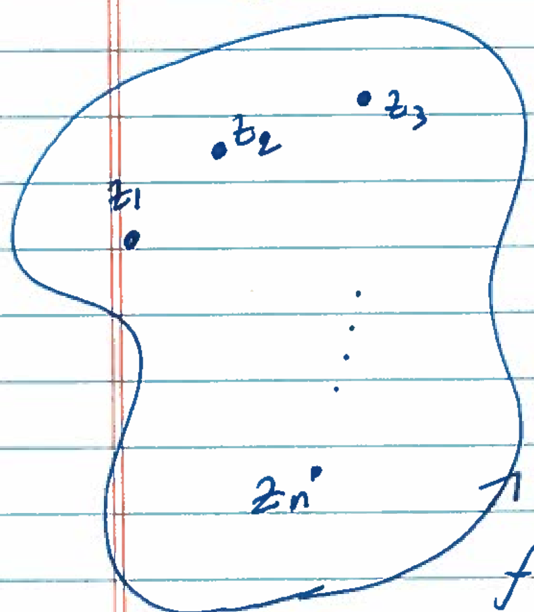
& Res

and $$\int_{C} \frac{\cos z \, dz}{(2z-\pi)^2} =$$

The Cauchy Residue Theorem

Let C be a ________ (with ______ orientation) and suppose $f(z)$ is ______ C and inside C ____________ $z_1, z_2, \ldots, z_n$ that are ________.

Then
$$\int_C f(z)\,dz =$$



PROOF:
Let $C_1, C_2, \ldots C_n$ be circles (negative orientation) with centers ________ with radii ________ ________. Then $f(z)$ is analytic on the ________ ________.

Let $B = C \cup C_1 \cup C_2 \cup \ldots \cup C_n$.
Then by the Cauchy-Goursat Theorem for multiply connected domains

$$\int_B f(z)\,dz =$$

$$\int_{B} f(z)\,dz =$$

So $$\int_{C} f(z)\,dz =$$

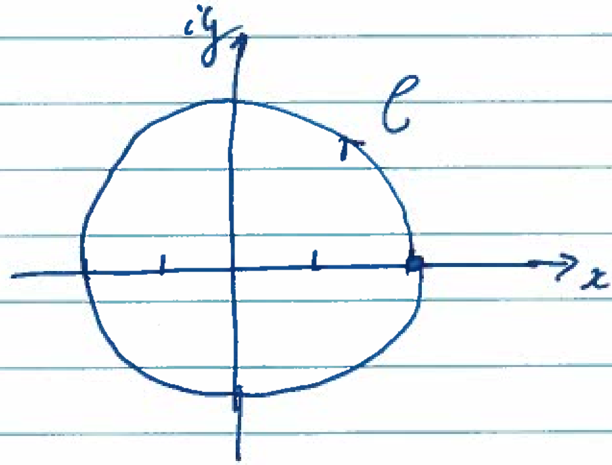
But each $-C_k$ is a simple closed contour that winds around the __________ so that

$$\int_{-C_k} f(z)\,dz =$$

Hence $$\int_{C} f(z)\,dz =$$

<u>Example</u> Find $\int_{\mathcal{C}} \frac{e^z}{z^2-1}\,dz$

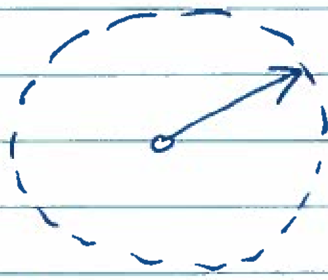
where $\mathcal{C}$ is the simple closed contour $|z|=2$ with positive orientation.



8-9).

Principal Part of a Function

Suppose $f(z)$ has an isolated singularity at $z = z_0$



There is an _ _ _
such that $f(z)$ is _ _ _ _
for

$f(z)$ has a Laurent expansion

$f(z) =$

for _ _ _ _ _.

The   is called
the _ _ _ _ _ _ _.

There are _ _ kinds of isolated singularities:

TYPE (I) Singularity

If the _ _ _ _ _ is _ _ _ then
the singularity is _ _ _ _.

Example The function $f(z) = \frac{\sin z}{z}$
has

TYPE II f has a __ __ __ of __ __ __ at __

There is an integer $m >$ __ such that

$b_n =$ for __ __ __.

The principal part of $f(z)$ has the form

In this case we say $f(z)$ has a __ __ __ __ at $z = z_0$

Example $f(z) = \dfrac{\cos z - 1}{z^4}$

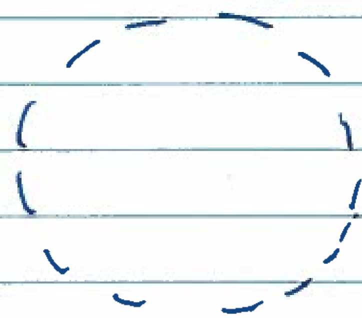
TYPE III f has an _ _ _ _ _ _ singularity at z_0

Example $f(z) = \exp\left(\frac{1}{z}\right)$.

Picards Theorem

Suppose $f(z)$ has an ________ singularity at z_0. Then an _____ neighborhood of z_0 $f(z)$ assumes _______________________________________.

Example Show that $\exp\left(\frac{1}{z}\right)=1$ for ~~oo~~ infinitely many z in any neighborhood of 0.



Residues at Poles

Recall $f(z)$ has a pole of order m at z_0 if it has a Laurent expansion of the form

$f(z) =$

<u>Proposition</u> Suppose $\phi(z)$ is analytic at z_0 and $\phi(z_0)$ ____ & m is a positive integer. Then

$$f(z) = \frac{\phi(z)}{(z-z_0)^m}$$

has a ____ of ____ __ at z_0 and

$$\operatorname*{Res}_{z=z_0} f(z) =$$

<u>Proof</u> $\phi(z)$ is analytic at z_0 so

$$\phi(z) =$$

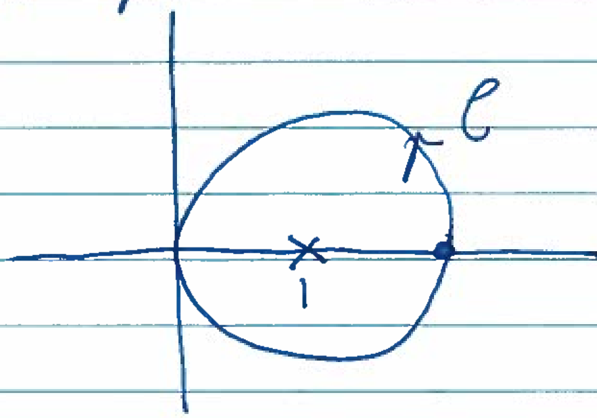
$$f(z) = \frac{\phi(z)}{(z-z_0)^m} =$$

$$=$$

So $z=z_0$ is a ____ of $f(z)$ of ____.

$$\operatorname*{Res}_{z=z_0} f(z) =$$

<u>Example</u> Let C be the simple circle $|z-1|=1$ with positive orientation.



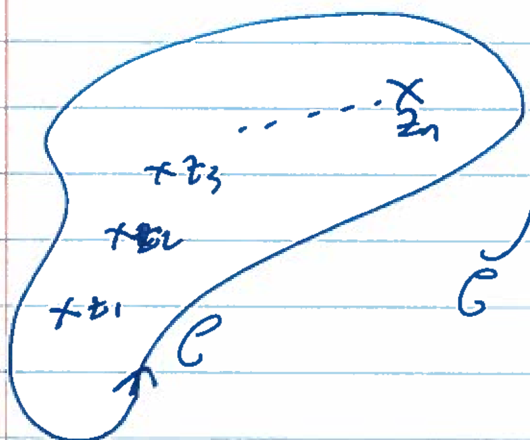
Find

$$\int_C \frac{z e^z}{(z-1)^2}\, dz$$

Residue at Infinity

Theorem: Let C be a simple closed contour with positive orientation. Suppose $f(z)$ is ______ ______ except for finitely many singularities $z_1, z_2, \ldots, z_n$ ______ ______.

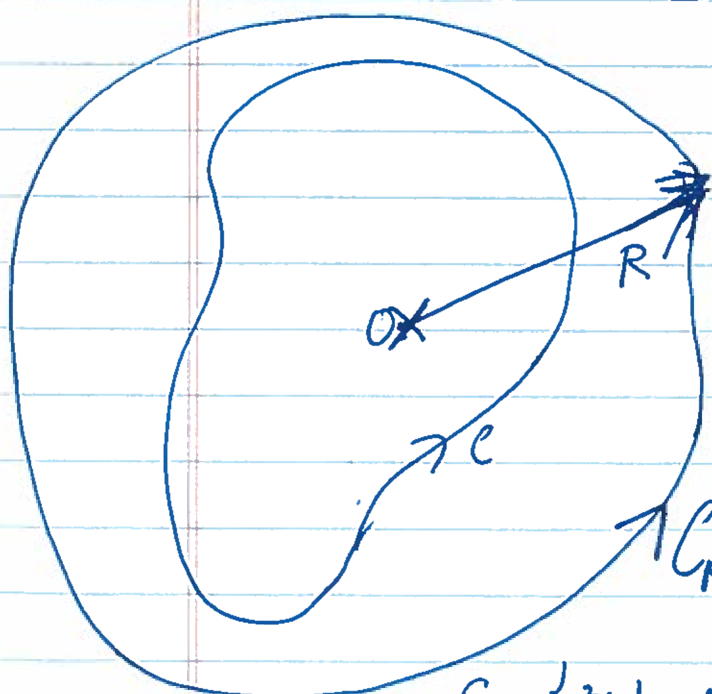
Then



$$\int_C f(z)\,dz =$$

PROOF: Choose $R > 0$ such that

$C \subset D(0, R)$. The $f(z)$ is analytic on the ______ ______ ______.



Hence

$$\int_C f(z)\,dz =$$

(by a Corollary to

$C_R : \{ z : |z| = R \}$

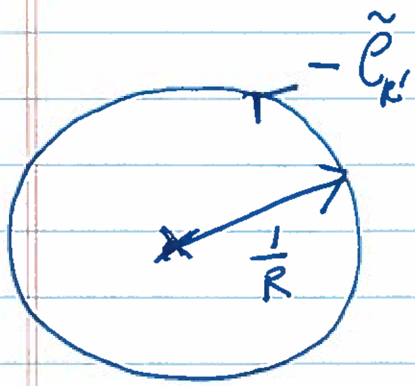
Let $C_R : \ z = Re^{it}, \quad 0 \leq t \leq 2\pi.$

Then $\quad dz =$

$$\int_{C_R} f(z)\,dz =$$

Let $g(z) = \frac{1}{z^2} f\left(\frac{1}{z}\right).$

Let $\tilde{C}_{R'} : \quad z = \frac{1}{R} e^{-it}, \quad 0 \leq t \leq 2\pi.$



Then

$$\int_{-\tilde{C}_{R'}} g(z)\,dz = -\int_{\tilde{C}_{R'}} g(z)\,dz$$

$=$

$g(z) = \frac{1}{z^2} f\left(\frac{1}{z}\right)$ is analytic for $0 < |z| \leq \frac{1}{R}$

since when $0 < |z| \leq \frac{1}{R}$, $\left|\frac{1}{z}\right| \geq$

and $f(z)$ is analytic for _ _ _.

The only singularity of $g(z)$ inside $-\tilde{C}_{R'}$ is

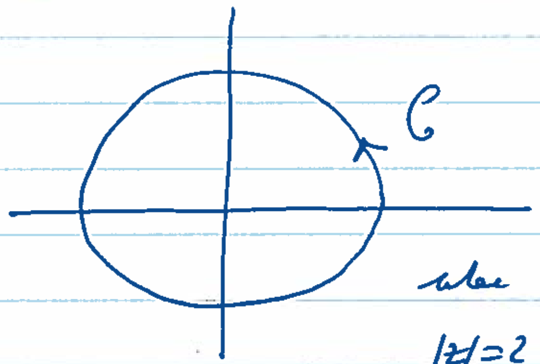
at $z =$ _ _ _. Hence

$$\int_{-\tilde{C}_{R'}} g(z)\,dz =$$

and therefore

$$\int_{C} f(z)\,dz =$$

<u>Example</u> (#3, p.247) Find



$$\int_C \frac{4z-5}{z(z-1)}\,dz$$

where C is the simple closed circle $|z|=2$ with +ve orientation

using (a) Cauchy Residue Theorem,
(b) Residue at infinity

Definition Suppose f is analytic at z_0 and z_0 is a zero of f ie $f(z_0) = 0$. We say z_0 is a **zero of order m** if

This means f has a Taylor expansion near $z = z_0$

$f(z) =$

$=$

$=$

where $g(z) =$

is _ ___ ___ and $g(z_0) =$

Theorem Let f be analytic at z_0. The following are equivalent

(i) f has a zero of order m at z_0.

(ii) There is a function g which is

Example The function $f(z) = \sin z$ has zeros at

$z =$

$f'(n\pi) =$

Hence all the zeros of f have _ _ _ _ .

We say these zeros are _ _ _ _ . For example,

$\sin z =$

$=$

and $z = 0$ is a _ _ _ _ _ .

Example $f(z) = 1 - \cos z$

$=$

$=$

$=$

and $z =$ _ _ is a _ _ _ _ _ .

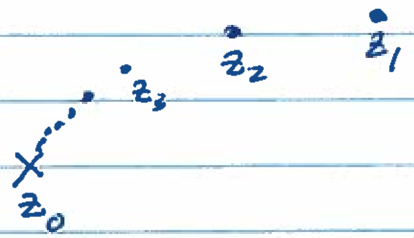
NOTATION. Let $f : D \to \mathbb{C}$. We write $f \equiv 0$ and say _ _ _ _ _ if

THEOREM: Let $D \subset \mathbb{C}$ be a domain (i.e _ _ _ _ _ _)

Suppose $f : D \to \mathbb{C}$ is analytic. Let

$$Z = Z(f) = \{ z \in D : \_ \_ \_ \}$$

If there is a $z_0 \in Z$ and a sequence $\{z_n\}_{n=1}^{\infty} \subset$ _ _ _ _ such that _ _ _ _ _ . Then $f \equiv 0$.



PROOF: Claim 1: $f^{(n)}(z_0) = 0$ for all $n = 0, 1, 2, \ldots$.

Proof of Claim: Suppose by way of contradiction that

$$f(z) = \sum_{n=n_0}^{\infty}$$

for $|z - z_0| < \delta$, some $\delta > 0$.

Let

$$g(z) = \sum_{n=n_0}^{\infty}$$

$$=$$

for $|z - z_0| < \delta$ and

$$g(z_0) =$$

$f(z) =$ __ __ __ __ __. for $|z - z_0| < \delta$

$z_n \in$ __ __ __ and $z_n \neq$ __ __ __ for $n \geq 1$

so that

__ __ $= f(z_n) =$ __ __ __ __ and $g(z_n) =$ for __ __ __

But $g(z)$ is analytic & hence __ __ __ __ for $|z - z_0| < \delta$

so $\lim_{n \to \infty} g(z_n) =$ and

$$g(z_0) =$$

which is a __ __ __ __. So Claim 1 __ __ __.

Hence $f^{(n)}(z_0) =$ for
and the set

$$E = \{ z \in D : f^{(n)}(z) = 0 \text{ for } \_\_\_ \}$$

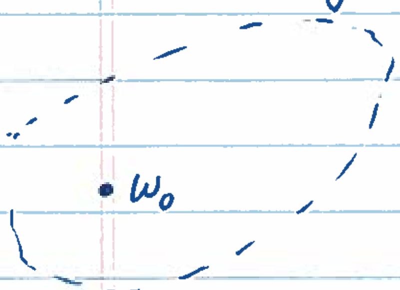
is ______ (since ______).
Let

$$E_n = \{ z \in D : f^{(n)}(z) = 0 \}.$$

Then each E_n is ______ since each $f^{(n)}$ is ______ and

$$E = \qquad \text{is a } \_\_\_\_\_\_ \text{ subset of } D.$$

<u>Claim 2</u> E is a open subset of ~~E~~ D.
<u>Proof of Claim</u>: Let $w_0 \in E$. Then $w_0 \in D$ and $f^{(n)}(w_0) =$
There is an $\varepsilon > 0$ such that



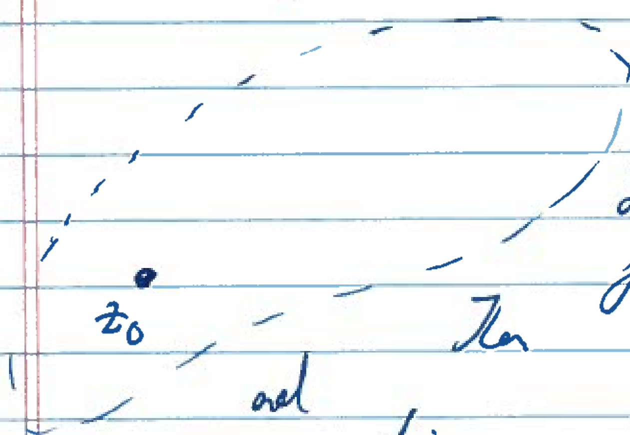
$$f(z) =$$

for all $|z - w_0| < \varepsilon$. Hence ______ and E is ______.

Since E is an nonempty, closed and open subset of D and D is ______, it follows that $E =$ ______; <u>ie</u> $f(z) =$

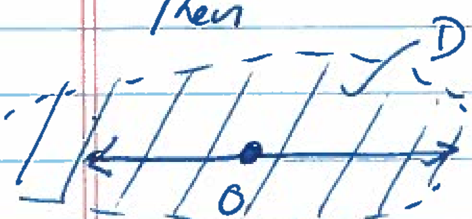
and $\square$

Corollary Let $D \subset \mathbb{C}$ be a domain.
Suppose $f: D \longrightarrow \mathbb{C}$ is analytic,
$D(z_0, r) \subset D$ and
$f(z) = 0$ for _ _ _ _ _.
Then
$f \equiv$ _ _ _.



PROOF.
$D(z_0, r) \subset$ _ _ _.
Let $z_n =$ _ _ _ _
for.
Then $\{z_n\} \subset$ _ _ _ _ . . . . ,
and
$\lim_{n \to \infty} z_n =$. The theorem
implies $f \equiv 0$. $\square$

Corollary Let $D \subset \mathbb{C}$ be a domain.
Suppose $f: D \longrightarrow \mathbb{C}$ is _ _ _ _ and
$\mathbb{R} \subset D$ and
$f(t) = 0$ for _ _ _ _.
Then



PROOF: $\mathbb{R} \subset$ _ _ _ _ _.
Let $z_n =$ for $n \geq 1$.
Then $\{z_n\} \subset$ _ _ _
$\lim_{n \to \infty} z_n =$

The theorem implies _ _ _ _ _. $\square$

Example PROVE $\sin^2 z + \cos^2 z = 1$ for all $z \in \mathbb{C}$.

PROOF:

Corollary Let $D \subset \mathbb{C}$ be a domain. Suppose $f: D \to \mathbb{C}$ is _ _ _ _ and $f \not\equiv 0$. Then _ _ _ _ zero of f is _ _ _ _ _ _.

PROOF: Suppose hypotheses hold. Suppose z_0 is a zero of f. Then $D(z_0, \delta) \subset D$ for some $\delta > 0$, since D is _ _ _ _. We want to show that there is a $0 < \delta < \delta_0$ such that

z_0

Suppose by way of contradiction that for each $0 < 1/n < \delta$

there is

Then

and

$$\lim_{n \to \infty} z_n$$

and $\{z_n\} \subset$

The theorem implies