Chapter 6 Residues & Poles

Definition: The point z_0 is called singular point of $f(z)$ if ____

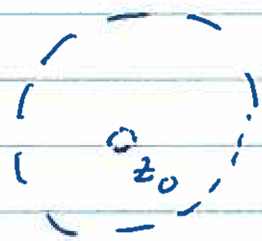
Examples

① $f(z) = \dfrac{\cos z}{z(z+1)}$

② $f(z) = \mathrm{Log}(z+3)$

Definition A singular point z_0 of $f(z)$ is <u>isolated</u> if ____

____.



Examples

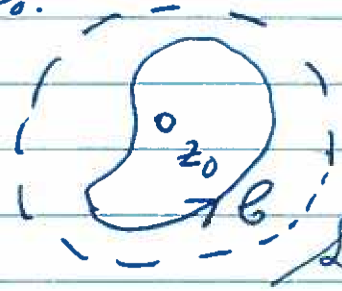
① $f(z) = \dfrac{\cos z}{z(z+1)}$

② $f(z) = \text{Log}\,(z+3)$.

③ $f(z) = \dfrac{1}{\sin(\pi/z)}$

Using Laurent Series to Compute a Contour Integral

Suppose $f(z)$ has an isolated singularity at z_0. Then for some ___ $f(z)$ is ____ for _______.



So $f(z)$ has a Laurent Expansion.

$f(t) =$

and

$b_n =$

Thus

$b_1 =$

and $\displaystyle\int_{C} f(z)\,dz =$

The coefficient b_1 is given a special name.
If z_0 is an _____ singularity of $f(z)$ then the residue of f at z_0 is ____
of _______ in the _______.

of _____ near ____.

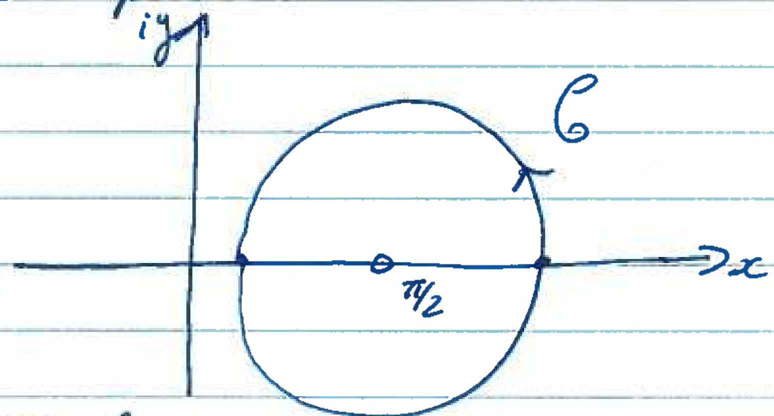
We write

for the residue of $f(z)$ at $z = z_0$.

<u>Example</u> Use Residues to compute

$$\int_C \frac{\cos z}{(2z-\pi)^2}\, dz$$

where C is the simple closed circle $|z - \pi/2| = 1$ with positive orientation.



The function $f(z) = \dfrac{\cos z}{(2z-\pi)^2}$ is analytic

everywhere except at ___ which _____. so

$$\int_C f(z)\, dz =$$

We need to find the coefficient of _____ in the Laurent expansion of ____ near ____.

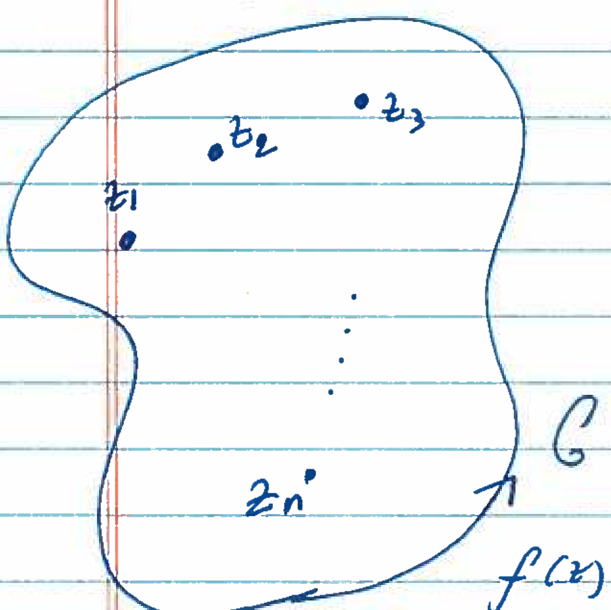
So Res

and $\displaystyle\int_C \frac{\cos z\, dz}{(2z-\pi)^2} =$

The Cauchy Residue Theorem

Let C be a ______________ (with ________ orientation) and suppose $f(z)$ is ________ C and inside C ______________________________ $z_1, z_2, \ldots, z_n$ that are ______________.

Then
$$\int_C f(z)\,dz =$$



PROOF:

Let $C_1, C_2, \ldots, C_n$ be circles (negative orientation) with centers ______________ with radii ______________ ______________. Then $f(z)$ is analytic on the ________ ______________________.

Let $\mathcal{B} = C \cup C_1 \cup C_2 \cup \ldots \cup C_n$.

Then by the Cauchy-Goursat Theorem for multiply connected domains

$$\int_{\mathcal{B}} f(z)\,dz =$$

$$\int_{B} f(z)\, dz =$$

So $$\int_{C} f(z)\, dz =$$

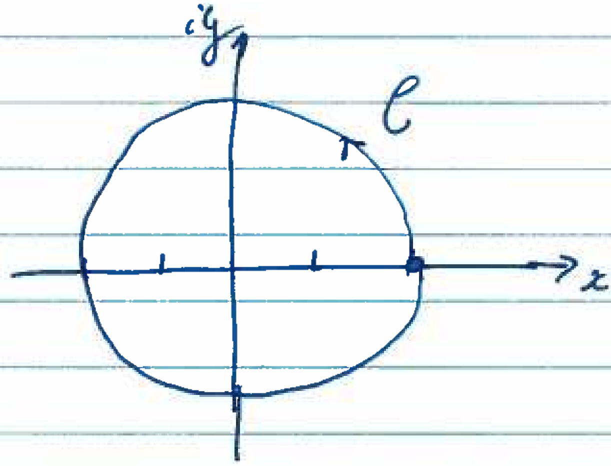
But each $-C_k$ is a simple closed contour that winds around the __________ so that

$$\int_{-C_k} f(z)\, dz =$$

Hence $$\int_{C} f(z)\, dz =$$

<u>Example</u> Find $\displaystyle\int_C \frac{e^z}{z^2-1}\,dz$

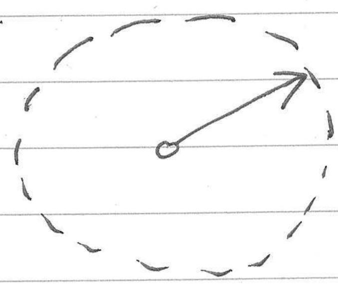
where C is the simple closed contour $|z|=2$ with positive orientation.



Principal Part of a Function

Suppose $f(z)$ has an isolated singularity at $z = z_0$

There is an ___ ___ ___
such that $f(z)$ is ___ ___ ___ ___
for



$f(z)$ has a Laurent expansion

$f(z) =$

for ___ ___ ___ ___ ___.

The   is called
the ___ ___ ___ ___ ___ ___ ___.

There are ___ kinds of isolated singularities:

TYPE (I)   Singularity

If the ___ ___ ___ ___ ___ ___ is ___ ___ ___ then
the singularity is ___ ___ ___ ___.

Example The function $f(z) = \frac{\sin z}{z}$
has

TYPE II f has a ___ of ___ at __

There is an integer $m >$ __ such that

$b_n =$ for ___.

The principal part of $f(z)$ has the form

In this case we say $f(z)$ has a ______

at $z = z_0$

Example $f(z) = \dfrac{\cos z - 1}{z^4}$

TYPE III f has an _ _ _ _ _ _ singularity at z_0

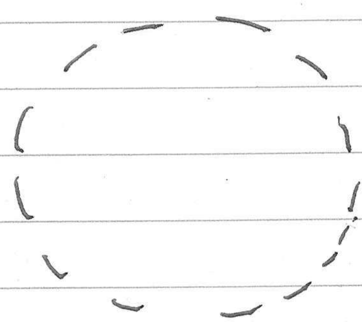
Example $f(z) = \exp\left(\frac{1}{z}\right)$.

Picards Theorem

Suppose $f(z)$ has an _______ singularity at z_0. Then on ____ neighborhood of z_0 $f(z)$ assumes ____________________

______________.

Example Show that $\exp\left(\frac{1}{z}\right) = 1$ for ~~a~~ infinitely many z in any neighborhood of 0.



Residues at Poles

Recall $f(z)$ has a pole of order m at z_0 if it has a Laurent expansion of the form

$f(z) =$

<u>Proposition</u> Suppose $\phi(z)$ is analytic at z_0 and $\phi(z_0)$ ____ & m is a positive integer. Then

$$f(z) = \frac{\phi(z)}{(z-z_0)^m}$$

has a ___ of ___ ___ at z_0 and

$$\operatorname*{Res}_{z=z_0} f(z) =$$

<u>Proof</u> $\phi(z)$ is analytic at z_0 so

$$\phi(z) =$$

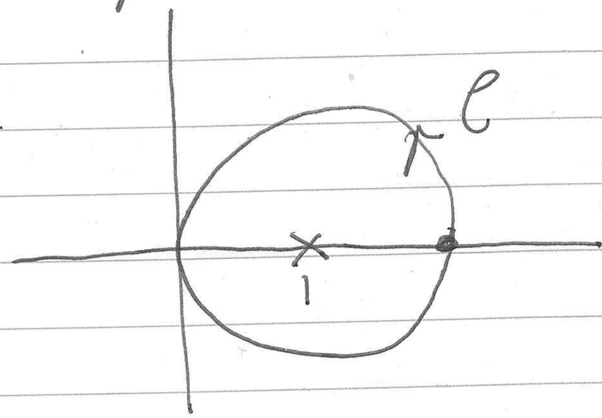
$$f(z) = \frac{\phi(z)}{(z-z_0)^m} =$$

$$=$$

So $z = z_0$ is a ___ ___ of $f(z)$ of ____

$$\operatorname*{Res}_{z=z_0} f(z) =$$

<u>Example</u> Let C be the simple circle $|z-1|=1$ with positive orientation.



Find

$$\int_C \frac{z e^z}{(z-1)^2}\, dz$$