Chapter 7 Applications of Residues

Evaluation of improper real integrals

Review Let $f:[0,\infty)\to\mathbb{R}$ be continuous.
We say the integral $\int_0^\infty f(x)\,dx$ converges
if

In which case we write $\int_0^\infty f(x)\,dx =$

Example Show that $\int_0^\infty e^{-x}\,dx$ converges and find its value.

Defn Let $f:\mathbb{R}\to\mathbb{R}$ be continuous. We say the integral $\int_{-\infty}^{\infty} f(x)\,dx$ converges if

<u>NOTE</u>: If both $\int_0^{\infty} f(x)\,dx$ & $\int_{-\infty}^{0} f(x)\,dx$ exist then

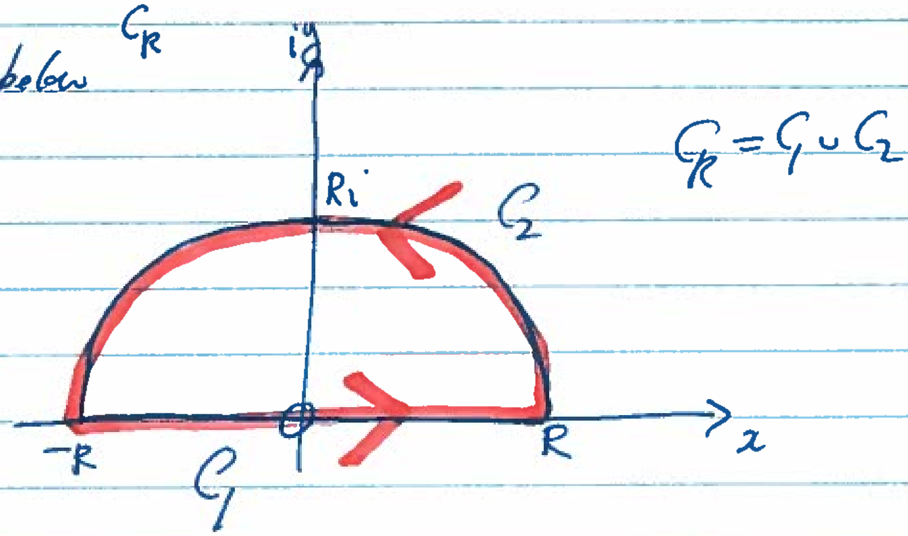
<u>WARNING</u> If $\lim_{R\to\infty} \int_{-R}^{R} f(x)\,dx$ exist then

For example, $\int_{-R}^{R} x\,dx =$

<u>THE IDEA</u> Consider

$$\int_{C_R} f(z)\,dz \quad \text{where } C_R \text{ is}$$

given below



C_1 : $z =$ $dz =$

$$\int_{C_1} f(z)\,dz =$$

C_2 : $z =$ $dz =$

$$\int_{C_2} f(z)\,dz =$$

$$\int_{C_R} f(z)\,dz =$$

$$=$$

We hope $\lim_{R\to\infty}$ so that

$$\int_{-\infty}^{\infty} f(x)\,dx =$$

Example Find $\displaystyle\int_{-\infty}^{\infty} \frac{x^2\,dx}{(x^2+1)(x^2+4)}$

Improper Integrals involving sin & cos

Example Find $\displaystyle\int_0^\infty \frac{\cos x}{x^2+1}\,dx$

Since $\cos x$ is an even function

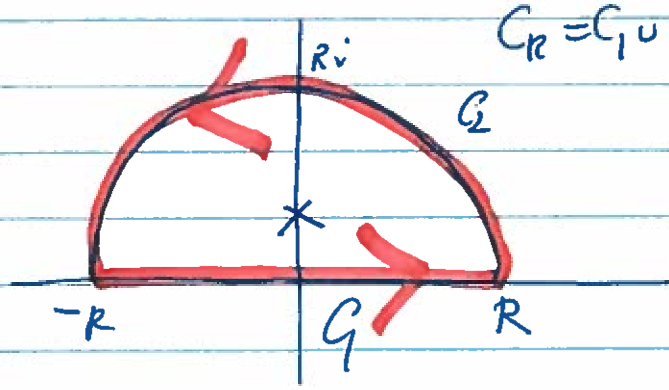
$$\int_0^\infty \frac{\cos x\,dx}{x^2+1} =$$

$$=$$

Let $\quad f(z) =$

Let C_R be the contour below where $R >$ ___

$C_R = C_1 \cup C_2$



$f(z)$ has singularities at $z =$ ___.

Let $z \in C_2$ where $z = x+iy$.

Then

$$|f(z)| =$$

Hence $\left| \int_{C_2} f(z)\, dz \right| \leqslant$

and so $\lim_{R \to \infty} \int_{C_2} f(z)\, dz =$

We have $\int_{C_R} f(z)\, dz =$

$=$

$\int_{C_R} f(z)\, dz =$

$\lim_{R \to \infty} \int_{C_R} f(z)\, dz =$

Hence $\int_{-\infty}^{\infty} \frac{\cos x}{x^2+1}\, dx =$

and $\int_{0}^{\infty} \frac{\cos x}{x^2+1}\, dx$