MAA 4402/5404 — FALL 2016 — EXAM I

NAME: __________

Instructions: Answer all questions. Show all necessary working and reasoning. Your work should be written in a proper and coherent manner, and in a way that any student in the class can follow your work. No calculators are allowed.

100% = 48 pts (30 students)

1. [5+5=10 pts]

(a) Complete the following properties of complex numbers:

(i) $|z_1 z_2| = |z_1|\,|z_2|$

(ii) $|\bar{z}| = |z|$

(iii) $|\mathrm{Re}(z)| \leq |z|$.

(iv) $|z_1 + z_2| \leq |z_1| + |z_2|$

(v) $|z_1 - z_2| \geq \big|\,|z_1| - |z_2|\,\big|$.

(b) Show clearly that

$$|\mathrm{Re}(2i - (\bar{z})^2 + z^3)| \leq 16 \text{ if } |z| \leq 2.$$

Suppose $|z| \leq 2$. Then

$|\mathrm{Re}(2i - (\bar{z})^2 + z^3)| \leq |2i + (-1)\bar{z}^2 + z^3|$ (by (iii))

$\leq |2i| + |(-1)\bar{z}^2| + |z^3|$ (by Δ ineq. (iv))

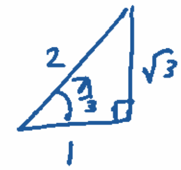
$= 2 + |\bar{z}|^2 + |z|^3$ (by (i))

$= 2 + |z|^2 + 8$ (by ii)

$= 2 + 4 + 8 = 14 \leq 16.$

Hence $|\mathrm{Re}(2i - (\bar{z})^2 + z^3)| \leq 16.$

$\tan\theta = -\frac{\sqrt{3}}{1}$



$\theta = -\frac{\pi}{3} + 2n\pi$

2. [2+3+3 = 8 pts]

(a) Find $\arg(1-\sqrt{3}i) = -\frac{\pi}{3} + 2n\pi \quad (n \in \mathbb{Z})$.

(b) Find $\operatorname{Arg}\left((1-\sqrt{3}i)^{100}\right)$

$$\arg\left((1-\sqrt{3}i)^{100}\right) = 100\arg(1-\sqrt{3}i) \quad (n \in \mathbb{Z})$$
$$= 100(-\pi/3) + 2n\pi$$
$$= -\frac{100}{3}\pi + 2n\pi = -33\tfrac{1}{3}\pi + 2n\pi.$$

$$\operatorname{Arg}\left((1-\sqrt{3}i)^{100}\right) = 34\pi - 33\tfrac{1}{3}\pi = \frac{2\pi}{3}.$$

(c) Find $\left|(2+i)^2(4+3i)\right| = |(2+i)|^2|4+3i|$

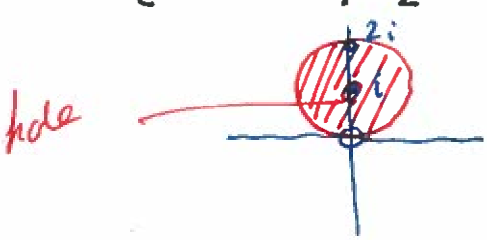
$$= (\sqrt{5})^2\sqrt{16+9} = 5\cdot\sqrt{25}$$
$$= 25$$

3. [4+4 = 8 pts]

~~Skech~~ Sketch the following sets of complex numbers:

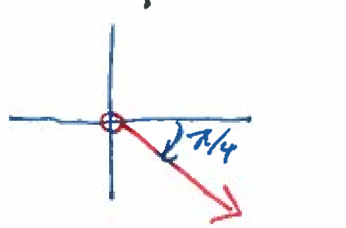
(a) $\left\{ z : \left|\frac{1}{z-i}\right| > 1 \right\}$ $\quad \left|\frac{1}{z-i}\right| = \frac{1}{|z-i|} > 1$

iff $|z-i| < 1$



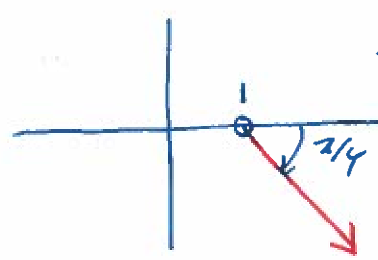
Circle center i, radius 1.

(b) $\left\{ z : \operatorname{Arg}(z-1) = -\frac{\pi}{4} \right\}$



$\operatorname{Arg}(z) = -\pi/4$

Answer



$\operatorname{Arg}(z-1) = -\pi/4$

4. [6 pts]
Sketch $S = \{ re^{i\theta} : 1 \le r \le 2 \text{ and } -\frac{\pi}{4} \le \theta \le 0 \}$

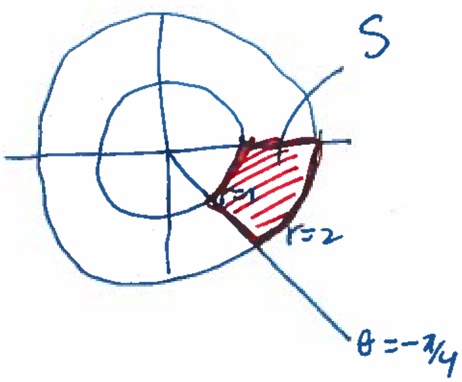
and the image of S under the mapping $f(z) = \frac{1}{z}$

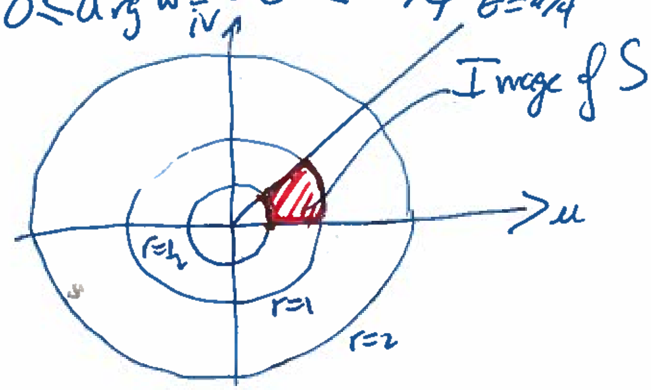
on two seperate diagrams. Let $z = re^{i\theta}$, $1 \le r \le 2$, $-\frac{\pi}{4} \le \theta \le 0$.

$w = \frac{1}{z} = (\frac{1}{r}) e^{-i\theta}$

$\frac{1}{2} \le |w| = \frac{1}{r} \le 1$,

$0 \le \arg w = -\theta \le \pi/4$





5. [2+6= 8 pts]
(a) Complete the definition: Let $z_0 \in \mathbb{C}$ and suppose $f: D'(z_0, r) \to \mathbb{C}$ for some $r > 0$. Let $w_0 \in \mathbb{C}$. We say the limit of $f(z)$ as z approaches z_0 is w_0 and write

$$\lim_{z \to z_0} f(z) = w_0 \quad \text{if}$$

for every $\varepsilon > 0$, there is a $\delta > 0$ such that

$|f(z) - w_0| < \varepsilon$ whenever $0 < |z - z_0| < \delta$.

(b) Use the formal definition of limit given in (a) to prove that

$$\lim_{z_0 \to z} Re(z) = Re(z_0).$$

PROOF: Let $\varepsilon > 0$ be any fixed real positive ~~integer~~ number. Suppose

$0 < |z - z_0| < \delta = \varepsilon.$

Then

$|Re(z) - Re(z_0)| = |Re(z - z_0)|$

$\leq |z - z_0| \quad (1(iii))$

$< \varepsilon.$ So

$|Re(z) - Re(z_0)| < \varepsilon$ if $0 < |z - z_0| < \varepsilon.$

Hence $\lim_{z \to z_0} Re(z) = Re(z_0).$ □

WORKING

Let $\varepsilon > 0$. We want

$|Re(z) - Re(z_0)| < \varepsilon.$

$|Re(z) - Re(z_0)|$

$= |Re(z - z_0)| \leq |z - z_0| < \delta$

Take $\delta = \varepsilon.$

6. [2+6 = 8pts]

(a) Complete the Definition:

Let $z_0 \in \mathbb{C}$ and suppose $f(z)$ is a complex-valued function defined on some neighborhood of z_0.

We say $f(z)$ is differentiable at z_0

if $\lim_{z \to z_0} \dfrac{f(z) - f(z_0)}{z - z_0}$ exists

If this limit exists then we write

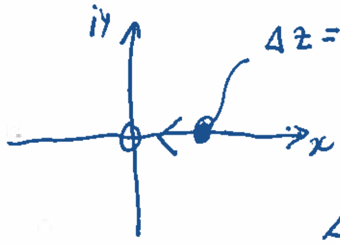
$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}.$$

(b) Let $f(z) = \operatorname{Im}(z)$ for $z \in \mathbb{C}$.
Show that $f'(z)$ does not exist at any point $z \in \mathbb{C}$ using (a) and <u>without</u> using the Cauchy-Riemann Equations.

Let $z \in \mathbb{C}$.

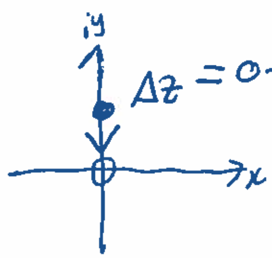
$$\lim_{\Delta z \to 0} \frac{\operatorname{Im}(z+\Delta z) - \operatorname{Im}(z)}{\Delta z} = \lim_{\Delta z \to 0} \frac{\operatorname{Im}(z) + \operatorname{Im}(\Delta z) - \operatorname{Im}(z)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\operatorname{Im}(\Delta z)}{\Delta z}$$



$\Delta z = \Delta x + i0$, $\operatorname{Im}(\Delta z) = 0$.

$$\lim_{\substack{\Delta z \to 0 \\ \Delta z \text{ on real axis}}} \frac{\operatorname{Im}(\Delta z)}{\Delta z} = \lim_{\Delta x \to 0} \frac{0}{\Delta x} = 0.$$



$\Delta z = 0 + i\Delta y$ $\operatorname{Im}(\Delta z) = \Delta y$

$$\lim_{\substack{\Delta z \to 0 \\ \Delta z \text{ on imag. axis}}} \frac{\operatorname{Im}(\Delta z)}{\Delta z} = \lim_{\Delta y \to 0} \frac{\Delta y}{i\Delta y} = \frac{1}{i} = -i$$

Since $0 \neq -i$. $\displaystyle\lim_{\Delta z \to 0} \frac{\operatorname{Im}(\Delta z)}{\Delta z}$ does not exist

and $f'(z)$ does not exist at any $z \in \mathbb{C}$.

7. BONUS. [2 pts]

(a) This is

A__________________

(b) When he was four years old his father fearing for his life in ________, moved his family to Arcueil. There things were hard & he wrote in a letter "We never had more than ______ a pound of ___ - and sometimes not even that."

(c) In 1816 he won the Grand Prix of the French Academy of Sciences for a work on ___ v ___. He achieved real fame however when he submitted a paper to the Institute solving one of F________'s claims on polygonal numbers made to Mersenne.