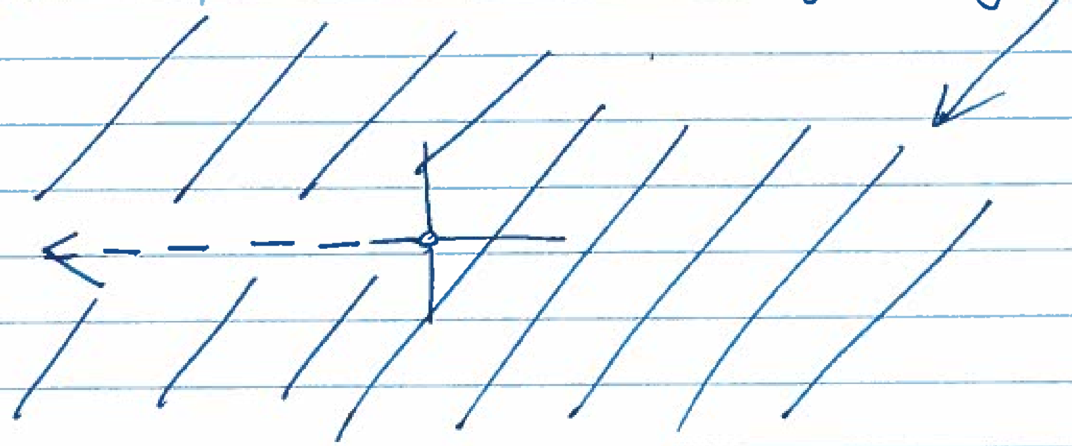
MAA 4402 – EXAM 2 – Solution.

1.

(a) Let $D = \{ z \in \mathbb{C} : z \neq 0 \ \& \ -\pi < \mathrm{Arg}\, z < \pi \}$.



$\mathrm{Log}(z)$ is analytic on D

(b) $$\mathrm{Log}\, z = \ln |z| + i \,\mathrm{Arg}\, z$$
$$= \ln r + i\theta$$

where $z = r e^{i\theta}$, $r > 0$ & $-\pi < \theta < \pi$.

Let $u = \ln r$, $v = \theta$.

$$\frac{\partial u}{\partial r} = \frac{1}{r}, \quad \frac{\partial v}{\partial \theta} = 1 \quad \text{so} \quad \frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$$

$$\frac{\partial v}{\partial r} = 0, \quad \frac{\partial u}{\partial \theta} = 0 \quad \text{so} \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}.$$

~~So Log z satisfies the Cauchy Riemann Eqns in polar form, so Log z is and~~ Since $\mathrm{Log}(z)$ is analytic,

Hence $$\frac{d}{dz} \mathrm{Log}\, z = e^{-i\theta}\left(\frac{\partial u}{\partial r} + i \frac{\partial v}{\partial r} \right)$$
$$= e^{-i\theta}\left(\frac{1}{r} + i\,0 \right) = \frac{1}{r e^{i\theta}} = \frac{1}{z}$$

for $z \in D$.

2.

$$\sin(z) = 2 \iff \frac{e^{iz} - e^{-iz}}{2i} = 2$$

$$\iff w - 1/w = 4i \quad \text{where} \quad w = e^{iz}$$

$$\iff w^2 - 1 = 4iw$$

$$\iff w^2 - 4iw - 1 = 0$$

$$\iff w = \frac{4i \pm \sqrt{-16+4}}{2}$$

$$= 2i \pm \sqrt{-4+1} = i(2 \pm \sqrt{3})$$

$$\iff iz = \log(i(2\pm\sqrt{3}))$$

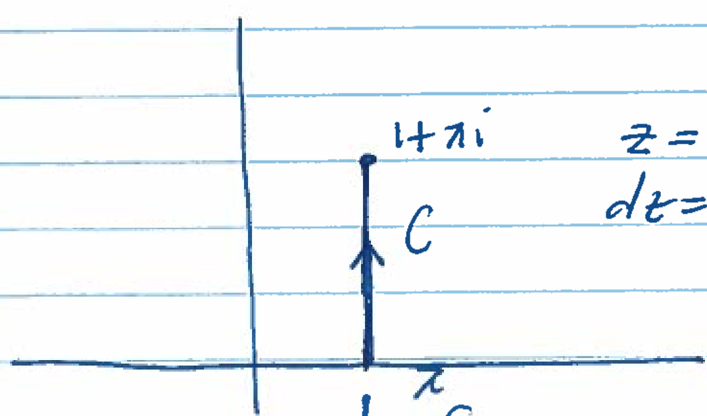
$$= \ln|i(2\pm\sqrt{3})| + i\arg(i(2\pm\sqrt{3}))$$

$$= \ln(2\pm\sqrt{3}) + \pi i/2 + (2\pi i n)$$

$$\iff z = (\pi/2 + 2\pi n) - i\ln(2\pm\sqrt{3})$$

where $n \in \mathbb{Z}$.

3.



$z = 1 + it, \quad 0 \leq t \leq \pi.$

$dz = i\,dt$

$$\int_C \exp(\bar{z})\,dz = \int_0^\pi \exp(1-it)\,i\,dt = e\int_0^\pi i e^{-it}\,dt$$

$$= e\; e^{-it}(-1)\Big]_0^\pi = -e(e^{-\pi i} - 1) = 2e.$$

4.

Let $|z| = 2$. Then

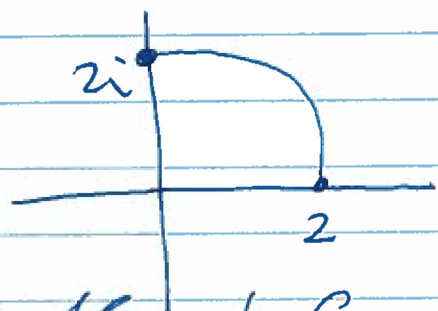
$$|\bar{z}+4| \leq |\bar{z}| + |4| = |z| + 4 = 6.$$

$$|z^2-1| \geq |z^2| - |1| = |z|^2 - 1 = 4-1 = 3.$$

So

$$\left|\frac{\bar{z}+4}{z^2-1}\right| = \frac{|\bar{z}+4|}{|z^2-1|} \leq \frac{6}{3} = 2 = M.$$

Length of $C = \frac{1}{4}(2\pi \cdot 2) = \pi = L$



Hence $\left|\int_C \frac{\bar{z}+4}{z^2-1}\,dz\right| \leq L \cdot M = 2\pi.$

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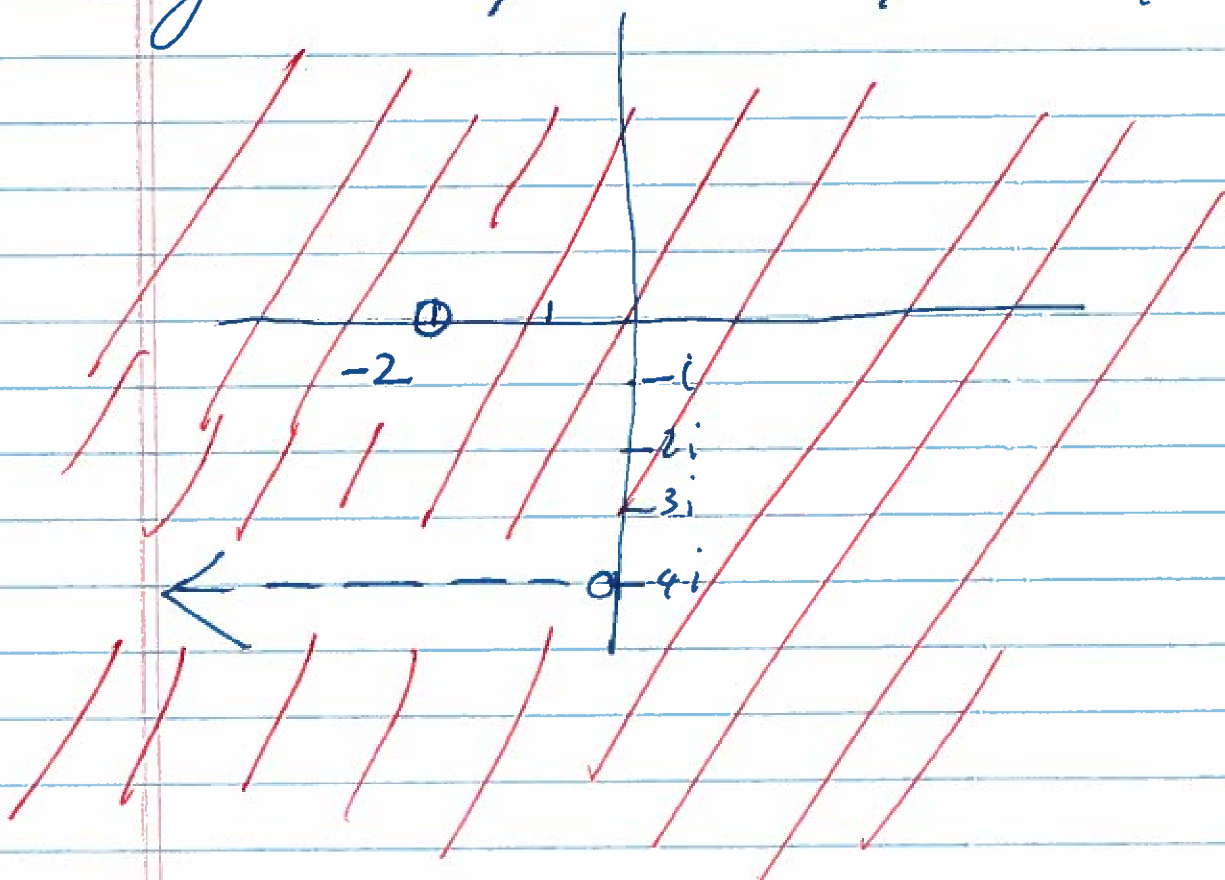
(i) $\text{Log}(z+4i)$ is analytic provided $z+4i \neq x+i0 \quad (x \le 0)$

i.e. $z \neq x-4i \quad (x \le 0)$.

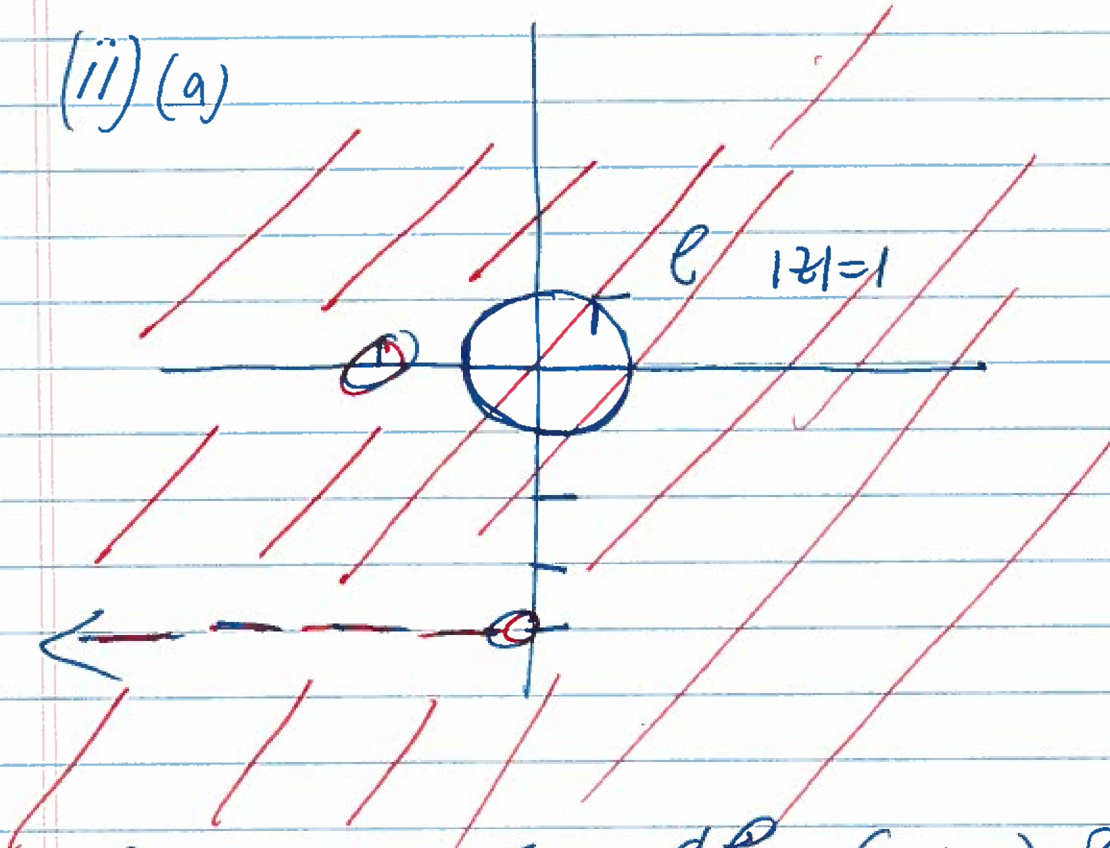
Hence

$$f(z) = \frac{\text{Log}(z+4i)}{(z+2)^2} \text{ is analytic}$$

for $z \neq -2$ & $z \neq x-4i \; (x \le 0)$



(ii) (a)

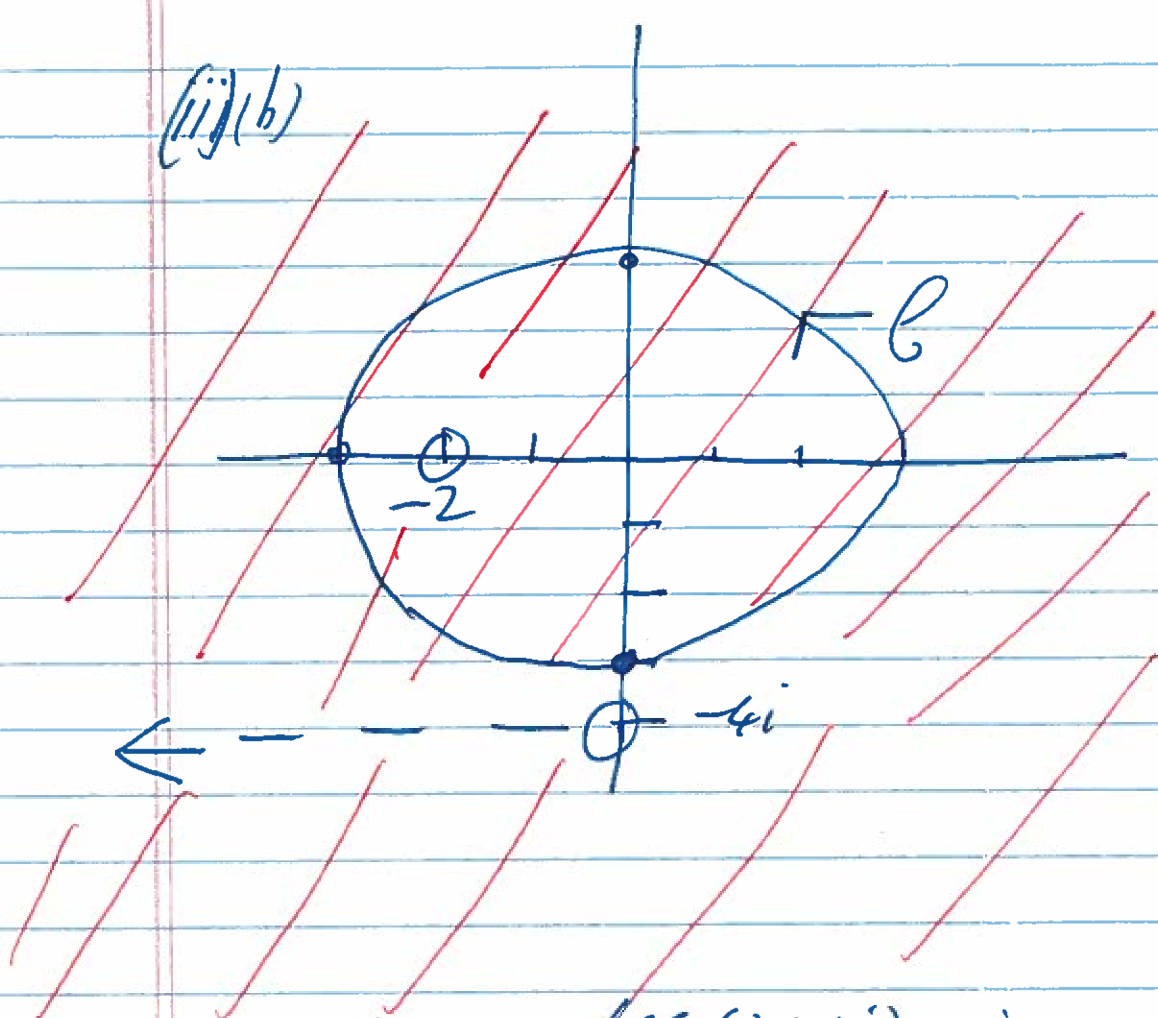


$f(z)$ is analytic on C $(|z|=1)$ & inside it

so by the Cauchy - Goursat theorem

$$\int_C f(z)\,dz = 0.$$

(ii)(b)



$g(z) = \text{Log}(z+4i)$ is analytic on C ($|z| = 3$) and inside it so by the Cauchy-Integral Formula for derivatives

$$\int_C f(z)\,dz = \int_C \frac{g(z)}{(z+2)^2}\,dz$$

$$= \frac{2\pi i}{1!}\, g'(-2)$$

since $z = -2$ is inside C.

(p.c)

$$g'(z) = \frac{1}{z+4i}$$

$$g'(-2) = \frac{1}{-2+4i} \quad \&$$

$$\int_C f(z)\,dz = \frac{2\pi i}{-2+4i} = \frac{\pi}{5}(2-i)$$