MAA 4402-5404 — FALL 2016 — EXAM 2

October 28, 2016

NAME: Solution

Instructions: all work should be written in a proper and coherent manner, and in a way that any student in the class can follow your work. Show all necessary working and reasoning.

[15 pts] 1. [TAKE HOME QUESTION — Due at LIT 408 by 4:30pm, Thursday October 27]

(a) Complete the following Theorem. Suppose

$$f(z) = u(x,y) + i\,v(x,y)$$

where $z = x+iy$ $(x, y \in \mathbb{R})$ is a complex-valued function defined in some open neighborhood of z_0 and suppose f is differentiable at $z_0 = x_0 + iy_0$ $(x_0, y_0 \in \mathbb{R})$ where $u, v \in \mathbb{R}$. Then the partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial v}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial y}$ exist at (x_0, y_0) and satisfy the Cauchy-Riemann Equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \& \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

at (x_0, y_0). Further,

$$f'(z_0) = \frac{\partial u}{\partial x}(x_0, y_0) + i\,\frac{\partial v}{\partial x}(x_0, y_0).$$

(b) Complete the following Theorem Suppose
$f(z) = u(x,y) + i\,v(x,y)$ where $z = x+iy$, $x,y,u,v \in \mathbb{R}$
is defined on an open neighborhood $D(z_0, r_0)$ of
$z_0 = x_0 + iy_0$, $x_0, y_0 \in \mathbb{R}$. Suppose the partial
derivatives $\frac{\partial u}{\partial x}, \frac{\partial v}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial y}$

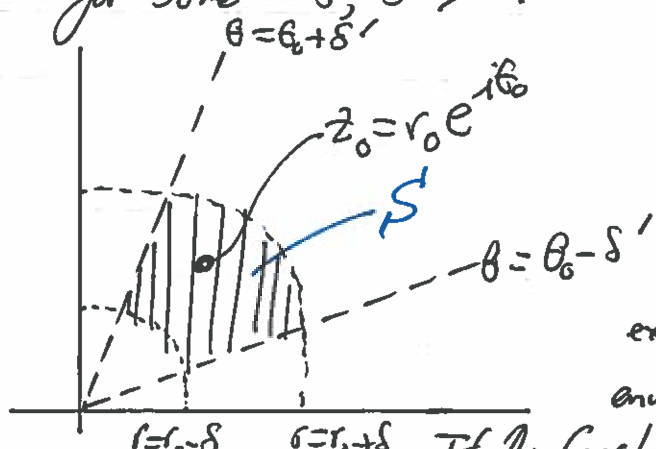
all exist on $D(z_0, r_0)$ and are continuous at (x_0, y_0).
Then if the Cauchy-Riemann Equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \& \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

hold at (x_0, y_0), then f is differentiable at z_0,
and

$$f'(z_0) = \frac{\partial u}{\partial x}(x_0,y_0) + i\,\frac{\partial v}{\partial x}(x_0,y_0).$$

(c) Complete the following Theorem Suppose
$f(re^{i\theta}) = u(r,\theta) + i\,v(r,\theta)$
for $0 < r_0 - \delta < r < r_0 + \delta$, $\theta_0 - \delta' < \theta < \theta_0 + \delta'$
for some $\delta, \delta' > 0$.



Suppose the partial derivatives $\frac{\partial u}{\partial r}, \frac{\partial v}{\partial r}, \frac{\partial u}{\partial \theta}$ & $\frac{\partial v}{\partial \theta}$ all

exist everywhere in S
and are continuous at (r_0, θ_0).
If the Cauchy-Riemann Equations (polar form)

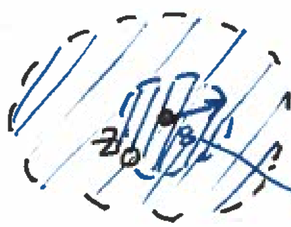
$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad \& \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

hold at (r_0, θ_0), then f is differentiable at z_0.

and

$$f'(z_0) = e^{-i\theta_0}\left(\frac{\partial u}{\partial r}(r_0,\theta_0) + i\frac{\partial v}{\partial r}(r_0,\theta_0)\right).$$

(d) Complete the Definition: Let $z_0 \in \mathbb{C}$ and suppose f is a complex-valued function defined on some open neighborhood of z_0. We say <u>f is analytic at z_0</u>



if f is differentiable at all points in some open neighborhood of z_0.

$D(z_0, \delta) = \{z : |z - z_0| < \delta\}$, some $\delta > 0$.

(e) Complete the <u>Proposition</u>: Suppose D, E are open subsets of $\mathbb{C}$, $f : D \to \mathbb{C}$, $g : E \to \mathbb{C}$ are analytic and $f(D) \subseteq E$. Then $g \circ f : D \to \mathbb{C}$ by $g \circ f(z) = g(f(z))$ is analytic on D, and

$$(g \circ f)'(z) = g'(f(z))\, f'(z)$$

for $z \in D$.

(f) Complete <u>Theorem</u>: Suppose $D \subset \mathbb{C}$ is a domain (i.e. a non-empty open and connected subset of $\mathbb{C}$). Suppose $f : D \to \mathbb{C}$ is analytic, and

$$f'(z) = 0 \quad \text{for all } z \in D.$$

Then f is a constant function, i.e. there exists a complex constant k such that $f(z) = k$ for all $z \in D$.

(g) Complete the Definition. Let $D \subset \mathbb{R}^2$ be a domain. A function $h: D \to \mathbb{R}$ is harmonic if the first and second partial derivatives of $h(x,y)$ are continuous on D and

$$\frac{\partial^2}{\partial x^2} h(x,y) + \frac{\partial^2}{\partial y^2} h(x,y) = 0$$

for all $(x,y) \in D$.

(h) Complete the Theorem. Let $D \subset \mathbb{C}$ be a domain and suppose $f: D \to \mathbb{C}$ is analytic, and

$$f(z) = u(x,y) + i\, v(x,y)$$

for $z = x+iy \in D$, $(x,y,\ u,v \in \mathbb{R})$. Then the functions $u(x,y)$ and $v(x,y)$ are harmonic on D.

(i) Complete the Definition. The exponential function $\exp(z)$ is defined by $\exp(z) = e^z := e^x e^{iy}$ for $z = x+iy$ where $x,y \in \mathbb{R}$.

(j) Let $z \in \mathbb{C}$, $w \in \mathbb{C}$ and $w \neq 0$. Then

$$\exp(z) = w$$

if and only if

$$z = \ln|w| + i\arg(w).$$

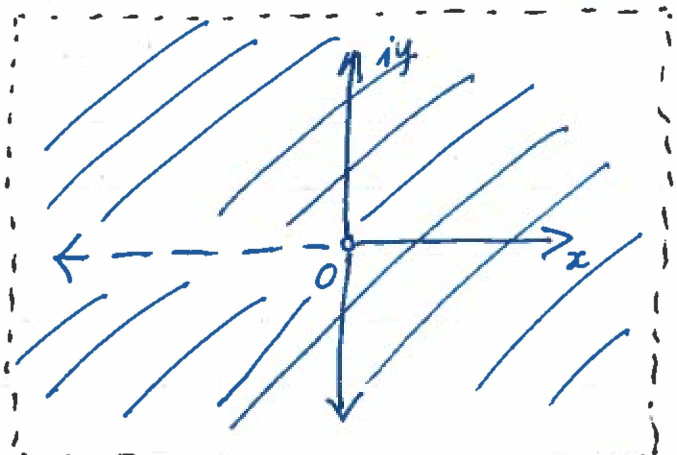
(k) Complete the <u>Definition</u> Let $z \neq 0$ the <u>principal value of the logarithm of z</u> is defined by

$$\mathrm{Log}(z) := \ln|z| + i\,\mathrm{Arg}(z)$$

(l) Complete <u>Theorem</u>

Let $D = \{ z \in \mathbb{C} : z \neq 0 \text{ and } -\pi < \mathrm{Arg}(z) < \pi \}$.

SKETCH D



Then $\mathrm{Log} : D \to \mathbb{C}$ is analytic and

$$\frac{d}{dz} \mathrm{Log}\, z = 1/z,$$

for $z \in D$.

(m) Complete the following <u>Definitions</u>: Let $z \in \mathbb{C}$.

Define

$$\cos(z) := \frac{1}{2}\left(e^{iz} + e^{-iz}\right),$$

$$\sin(z) := \frac{1}{2i}\left(e^{iz} - e^{-iz}\right)$$

(n) Complete the following Properties

Let $z = x+iy$, where $x, y \in \mathbb{R}$. Then

$$|\sin z|^2 = (\sin x)^2 + (\sinh y)^2,$$

and

$$|\cos z|^2 = (\cos x)^2 + (\sinh y)^2.$$

(o) Complete the following Definition. Let $z \in \mathbb{C}$ suppose $z \neq 0$, $c \in \mathbb{C}$. The power function z^c is defined by

$$z^c := \exp(c \log z).$$

This is a multi-valued function. The principal value of z^c is denoted by $P.V.\, z^c$ and is defined by

$$P.V.\, z^c := \exp(c \operatorname{Log} z).$$

2. [5 + 10 + 5 = 20 pts] Explain your reasoning in each part.

(a) What does the Theorem in Qu. 1(a) imply about the function $f(z) = \bar{z}$?

(b) Let $f(z) = z \operatorname{Im} z$.
Determine where $f'(z)$ exists and find its value

(c) Where is the function $f(z) = z \operatorname{Im} z$ analytic ?

(a) $f(z) = \bar{z} = x - iy, \quad z = x + iy$.
Let $u = x, \; v = -y$.
$\frac{\partial u}{\partial x} = 1 \neq -1 = \frac{\partial v}{\partial y}$ for all (x, y).
The Cauchy-Riemann Eqns hold nowhere so f is d'ble nowhere.

(b) Let $z = x + iy, \quad x, y \in \mathbb{R}$.
Then
$$f(z) = \cancel{z \operatorname{Im} z} = (x + iy) y = xy - iy^2.$$
Let
$$u = xy, \quad v = -y^2.$$
$$\frac{\partial u}{\partial x} = y, \qquad \frac{\partial v}{\partial y} = -2y$$
$$\frac{\partial u}{\partial y} = x, \qquad \frac{\partial v}{\partial x} = 0.$$
These partial derivatives exist and are continuous for all (x, y).

$$C\text{-}R: \begin{cases} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \end{cases} \Leftrightarrow \begin{cases} y = -2y \\ x = 0 \end{cases} \Leftrightarrow \begin{cases} x = 0 \\ y = 0 \end{cases}$$

The Cauchy-Riemann Eqns only hold at $(0,0)$.
So f is only d'ble at $z=0$ &

$$f'(0) = \frac{\partial u}{\partial x}(0,0) + i\frac{\partial v}{\partial x}(0,0) = 0 + i0 = 0.$$

(c) f is only d'ble at 0.
It is not d'ble on any open nbd of 0
So f is analytic nowhere.

12.5/4 3. [5 + 10 = 15 pts]

(a) Give an example of a $z \in \mathbb{C}$ that satisfies

$$\mathrm{Log}\, z^2 \neq 2\,\mathrm{Log}\, z.$$

Explain reasoning.

(b) Sketch the domain on which

$$f(z) = \frac{\mathrm{Log}(z+i)}{z^2+1}.$$

is analytic. Explain your reasoning.

(a) Let $z = e^{3\pi i/4}$.

Then

$$z^2 = e^{3\pi i/2} = -i$$

$$\mathrm{Log}(z) = \ln|e^{3\pi i/4}| + i\,\mathrm{Arg}(e^{3\pi i/4})$$
$$= \ln 1 + i\,3\pi/4$$
$$= 3\pi i/4.$$

$$\mathrm{Log}(z)^2 = \mathrm{Log}(-i) = \ln|-i| + i\,\mathrm{Arg}(-i)$$
$$= \ln 1 - \pi i/2 = -\pi i/2.$$

$$\mathrm{Log}(z^2) = -\pi i/2 \neq 3\pi i/2 = 2\,\mathrm{Log}(z).$$

(b) $\mathrm{Log}(z+i)$ is analytic for $z+i \neq x+i0$ where $x \leq 0$

i.e. $z \neq x - i$ where $x \leq 0$.

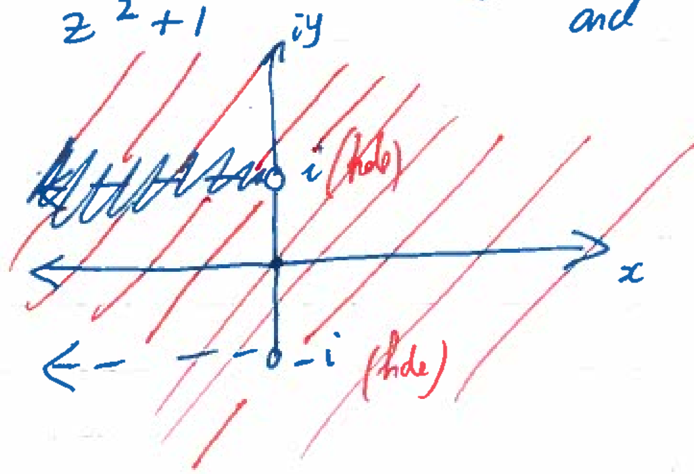
(since $z+i$ is entire).

$z^2+1 = 0$ when $z = \pm i$

& $\frac{1}{z^2+1}$ (rational fcn) is analytic for $z \neq \pm i$

Hence

$\dfrac{\text{Log}(z+i)}{z^2+1}$ is analytic for z satisfying $z \neq x-i,\ x \leq 0$ and $z \neq i$, & $z \neq -i$.



Region on which function is analytic.

4. [10 + 10 = 20 pts]

DO ONLY TWO PARTS

12.5?. (a) Solve $\sin z = i$

(b) Find P.V. $(1-i)^{4i}$

(c) Show that $|\exp(z^2)| \leq \exp(|z|^2)$ for all $z \in \mathbb{C}$.

$\sin z = i \iff \frac{1}{2i}\left(\exp(iz) - \exp(-iz)\right) = i$

$\iff w - \frac{1}{w} = -2$ where $w = \exp(iz)$

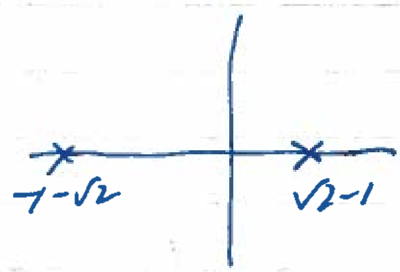
$w^2 + 2w - 1 = 0$

$w^2 + 2w + 1 = 2$

$(w+1)^2 = 2$

$w + 1 = \pm\sqrt{2}$

$w = -1 \pm \sqrt{2}$



$\exp(iz) = -1 \pm \sqrt{2}$

$iz = \ln|-1 \pm \sqrt{2}| + i \arg(-1 \pm \sqrt{2})$

$= \ln(\sqrt{2} \mp 1) + i \arg(-1 \pm \sqrt{2})$

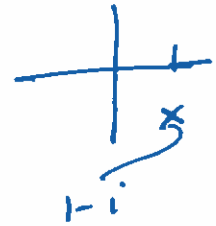
$iz = \ln(\sqrt{2}-1) + i \arg(\sqrt{2}-1) = \ln(\sqrt{2}-1) + 2\pi n i$

or $iz = \ln(\sqrt{2}+1) + i \arg(-\sqrt{2}-1) = \ln(\sqrt{2}+1) + (-\pi + 2n\pi)i$

So $z = 2\pi n - i\ln(\sqrt{2}-1)$, or

$(2n-1)\pi - i\ln(\sqrt{2}+1)$ for $n \in \mathbb{Z}$.

(b) $$\text{P.V.}\,(1-i)^{4i} = \exp(4i\,\mathrm{Log}(1-i))$$



$$= \exp 4i(\ln|1-i| + i\,\mathrm{Arg}(1-i)))$$

$$= \exp(4i(\ln\sqrt{2} - \pi i/4))$$

$$= \exp(+\pi + 2i\ln 2)$$

$$= e^{\pi}(\cos(2\ln 2) + i\sin(2\ln 2))$$

(c) Let $z = x+iy$, where $x, y \in \mathbb{R}$.
Then

$$z^2 = (x+iy)^2 = x^2 + 2xiy - y^2$$
$$= x^2 - y^2 + 2xyi$$
$$|z|^2 = x^2+y^2.$$

$$|\exp(z^2)| = |e^{x^2-y^2}e^{2xyi}| = e^{x^2-y^2} \le e^{x^2+y^2} = \exp|z|^2$$

since $-y^2 \le y^2$ & $x^2-y^2 \le x^2+y^2$.

5. [10 + 10 = 20 pts]

DO ONLY TWO PARTS

(a) Prove that $Log(z)$ is analytic on the domain $D = \{z \in \mathbb{C} : z \neq 0 \text{ and } -\pi < Arg\, z < \pi\}$.

(b) Suppose $f(z) = u(x,y) + iv(x,y)$ and its conjugate $\overline{f(z)} = u(x,y) - iv(x,y)$ are both analytic on a domain D. Prove that $f(z)$ must be a constant function on D. [Here $z = x+iy$, $x, y, u, v \in \mathbb{R}$].

(c) Prove that $\exp(z)$ is an entire function and $\frac{d}{dz}\exp(z) = \exp(z)$ for all z.

(d) Suppose $f(z) = u(x,y) + iv(x,y)$ is analytic on a domain D (where $z = x+iy$, $x, y \in \mathbb{R}$). Prove that the functions

$$U(x,y) = e^{u(x,y)}\cos(v(x,y)),$$
$$V(x,y) = e^{u(x,y)}\sin(v(x,y)),$$

are harmonic on D.

(a) $z \in D = \{z : z \neq 0 \ \& \ -\pi < \operatorname{Arg} z < \pi\}$
if and only if $z = re^{i\theta}$ where $r > 0$ & $-\pi < \theta < \pi$.

Let $z = re^{i\theta}$ where $r>0$ & $-\pi < \theta < \pi$.
Then $\operatorname{Log} z = \ln |z| + i \operatorname{Arg} z$
$= \ln r + i\theta$.
Let $u = \ln r$, $v = \theta$.
The partial derivatives
$$\frac{\partial u}{\partial r} = \frac{1}{r}, \quad \frac{\partial u}{\partial \theta} = 0, \quad \frac{\partial v}{\partial r} = 0, \quad \frac{\partial v}{\partial \theta} = 1,$$
exist and are continuous for $r>0, -\pi < \theta < \pi$.
$$\frac{\partial u}{\partial r} = \frac{1}{r} \cdot 1 = \frac{1}{r}\frac{\partial v}{\partial \theta},$$
$$\frac{\partial v}{\partial r} = 0 = \frac{-1}{r} \cdot 0 = \frac{-1}{r}\frac{\partial u}{\partial \theta} \quad \text{for } r>0, \ -\pi<\theta<\pi,$$
& the Cauchy-Riemann Equations (polar form) hold on this region
Therefore $\operatorname{Log} z$ is analytic on D.

(b) Suppose $f(z) = u(x,y) + i\, v(x,y)$,
and $\overline{f(z)} = u(x,y) - i\, v(x,y)$
are both analytic on a domain D.
Since f is analytic on D,
$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \& \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \text{on D.}$$

Since $\overline{f}$ is analytic on D,
$$\frac{\partial u}{\partial x} = \frac{\partial (-v)}{\partial y} \quad \& \quad \frac{\partial u}{\partial y} = -\frac{\partial (-v)}{\partial x} \quad \text{on D.}$$

So $\frac{\partial v}{\partial y} = -\frac{\partial v}{\partial y}$ & $\frac{\partial v}{\partial x} = -\frac{\partial v}{\partial x}$ on D.

Thus $\frac{\partial v}{\partial y} = \frac{\partial v}{\partial x} = 0$ on D.

So $f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$

$= \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x} = 0 + i0 = 0$

for $z \in D$. $f'(z) = 0$ for all ~~z∈~~ D.
This implies f is a constant function D
by the Theorem in 1(f).

(c) $\exp(z) = e^x e^{iy}$

$= e^x \cos y + i\, e^x \sin y$

for $z = x+iy \in \mathbb{C}$ $(x, y \in \mathbb{R})$.
Let $u = e^x \cos y$, $v = e^x \sin y$.
The partial derivatives

$\frac{\partial u}{\partial x} = e^x \cos y$, $\frac{\partial v}{\partial x} = e^x \sin y$,

$\frac{\partial u}{\partial y} = -e^x \sin y$, $\frac{\partial v}{\partial y} = e^x \cos y$

exist and are continuous for all x, y.

$\frac{\partial u}{\partial x} = e^x \cos y = \frac{\partial v}{\partial y}$, &

$\frac{\partial u}{\partial y} = -e^x \sin y = -\frac{\partial v}{\partial x}$ for all x, y.

hence the Cauchy-Riemann Equations hold everywhere & $\exp(z)$ is entire &

$$\frac{d}{dz}\exp(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$= e^x \cos y + i e^x \sin y$$

$$= \exp(z)$$

for all $z \in \mathbb{C}$.

(d) Suppose $f(z) = u(x,y) + i\,v(x,y)$ is analytic on a domain D. Then

$$g(z) = \exp(f(z)) \quad \text{is analytic on } D$$

since exp is entire.

$$g(z) = \exp(u(x,y) + i\,v(x,y))$$

$$= \exp(u(x,y))\,(\cos(v(x,y)) + i\sin(v(x,y)))$$

$$= \exp(u(x,y))\cos((v(x,y)) + i\exp(u(x,y))\sin(v(x,y)).$$

Hence

$$U(x,y) = \exp(u(x,y))\cos(v(x,y)), \ \&$$

$$V(x,y) = \exp(u(x,y))\sin(v(x,y))$$

are harmonic on D by the Theorem in 1(h).