Chapter 1 Linear Equations in Linear Algebra

1.1 Systems of Linear Equations

Defn A linear equation in the variables $x_1, x_2, \ldots, x_n$
has the form

where

Examples

(1) The equation
$$5x + 3y = 1$$
is

(2) $$5x_1 + 2x_2 - x_3 + 4x_4 = 11$$
is

(3) $$3x + 4y + 5z^2 = 2$$
is

Defn A system of linear equations (or a)
is a

Example

(1) $$5x_1 + 2x_2 = 9$$
$$x_1 - x_2 = -1$$

is a

(2) $$x_1 + 2x_2 - 3x_3 = -5$$
$$x_1 + x_2 - x_3 = 4$$

is

Def$^{\underline{n}}$ A solution of a system is a

Example

(1) $(1, 2)$ is a solution of the system in (1) ~~[illegible]~~ since when

(2) Similarly $(13, -9, 0)$ is a solution of the system in (2) since

Def$^{\underline{n}}$ The set of all solutions of a linear system is called the ______________.

Example

(1) We have seen that (,) is a solution to the system

$$5x_1 + 2x_2 = 9,$$
$$x_1 - x_2 = -1.$$

Is there any other solution?
Suppose (x_1, x_2) is a solution. Then

and
so that
Hence (x_1, x_2)

The solution set =

(2) We have seen that (, ,) is a solution to the linear system

$$x_1 + 2x_2 - 3x_3 = -5$$
$$x_1 + x_2 - x_3 = 4$$

Are there any other solutions? ___,
for example (, ,) is
It can be shown that there are _______ many solutions. In fact, it can be shown that the solution set

$$S = \{ (\quad , \quad , \quad) : \qquad \}.$$

This means that $x_1 =$, $x_2 =$, $x_3 =$
is a _ _ _ _ _ _ _ _ _ _ and any solution
has this _ _ _ _ . For instance letting $t =$ _ _ _
we get the solution $(13, -9, 0)$.
Letting $t =$ _ _ _ we get the solution
$(\quad , \quad , \quad) = (\quad , \quad , \quad)$.

(3) The linear system

$$x_1 + x_2 = 1$$
$$x_1 + x_2 = 2$$

clear has _ _ _ _ _ _ _ _ _.
The solution set =

<u>NOTE:</u> We have seen that a linear system has either
(1)
(2)
or (3)
It can be shown that there are <u>NO</u> other possibilities.

<u>Defn</u> Two linear systems are <u>equivalent</u>
if

<u>Example</u> The linear systems

$$\begin{cases} 5x_1 + 2x_2 = 9 \\ x_1 - x_2 = 7 \end{cases} \quad \text{and} \quad \begin{cases} x_1 + x_2 = 3 \\ 2x_1 + x_2 = 4 \end{cases}$$

have the same solution set namely $\{ \qquad \}$.
We say that these systems are _ _ _ _ _ _ _ and

write $\begin{cases} 5x_1 + 2x_2 = 9 \\ x_1 - x_2 = 1 \end{cases}$ ..... $\begin{cases} x_1 + x_2 = 3 \\ 2x_1 + x_2 = 4. \end{cases}$

NOTE The symbol " $\Leftrightarrow$ " means " .... ".

OPERATIONS THAT DO NOT CHANGE THE SOLUTION SET

(1) Multiply both sides of an equation by a ______.

Example

$$\begin{matrix} 5x_1 + 2x_2 = 9 \\ x_1 - x_2 = -1 \end{matrix} \quad \Leftrightarrow \quad \begin{matrix} 5x_1 + 2x_2 = 9 \\ \end{matrix}$$

(2) Multiply one equation by ___ and ___ to ________.

NOTE Here the _____ equation gets ______.

Example

$$\begin{matrix} 5x_1 + 2x_2 = 9 \\ x_1 - x_2 = -1 \end{matrix} \quad \Leftrightarrow \quad \begin{matrix} \\ x_1 - x_2 = -1 \end{matrix}$$

(3) ______ two equations.

Example

$$\begin{matrix} 5x_1 + 2x_2 = 9 \\ x_1 - x_2 = -1 \end{matrix} \quad \Leftrightarrow$$

MATRIX NOTATION

A linear system can be represented more compactly as a matrix (a ____________ of ______).

Example The linear system

$$x_1 - 2x_2 + x_3 = 0$$
$$2x_2 - 8x_3 = 8$$
$$-4x_1 + 5x_2 + 9x_3 = -9$$

can be represented by the ________ matrix

SOLVING LINEAR SYSTEMS

The basic strategy is to use the __________ to replace the system by an ______________ which is ____________.

EXAMPLE Solve the system above:

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ -4x_1 + 5x_2 + 9x_3 &= -9 \end{aligned} \qquad \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{array}\right]$$

$$\Longleftrightarrow$$

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ &= \end{aligned} \qquad \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & & & \end{array}\right]$$

$$\Longleftrightarrow$$

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ & \\ -3x_2 + 13x_3 &= -9 \end{aligned} \qquad \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ & & & \\ 0 & -3 & 13 & -9 \end{array}\right]$$

This system is ______ form

$$\Longleftrightarrow$$

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ x_2 - 4x_3 &= 4 \\ & \end{aligned} \qquad \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & & & \end{array}\right]$$

$$\Longleftrightarrow$$

$$\begin{aligned} x_1 - 2x_2 \quad &= \\ x_2 \quad &= \\ x_3 &= 3 \end{aligned} \qquad \left[\begin{array}{ccc|c} 1 & -2 & & \\ 0 & 1 & & \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$$\Longleftrightarrow$$

$$\begin{aligned} x_1 \quad &= \\ x_2 \quad &= \\ x_3 &= 3 \end{aligned} \qquad \left[\begin{array}{ccc|c} 1 & 0 & 0 & \\ 0 & 1 & 0 & \\ 0 & 0 & 1 & 3 \end{array}\right]$$

The system has ____ solution $(x_1, x_2, x_3) =$

Can we check our answer?

NOTE: The operations we used on the augmented matrix are called __________ __________ operations.

ELEMENTARY ROW OPERATIONS

(1) Add a multiple of ____ to ____

NOTATION: (only changes)

(2) ________ two rows.

NOTATION:

(3) Multiply one ____ by ________.

NOTATION:

Defn A linear system is consistent if

EXISTENCE & UNIQUENESS QUESTIONS

Suppose we are given a linear system.

(1) Is the system ________? In other words does it have ________? Or does it have ___ ____ (ie _____).

(2) If the system is consistent is the solution ____? ie Is there just ___ ______ or are there ________?

Example

(1) We have seen that the linear system

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ -4x_1 + 5x_2 + 9x_3 &= -9 \end{aligned}$$

is ________. In fact it has the ________ solution (, ,).
The ________ is immediate once we see the system in ________ form.

EXAMPLE

(2) Is the following ~~8x~~ system consistent?

$$\begin{aligned} 3x_2 - 6x_3 &= 8 \\ x_1 - 2x_2 + 3x_3 &= -1 \\ 5x_1 - 7x_2 + 9x_3 &= 0 \end{aligned}$$

$$\Leftrightarrow \left[\begin{array}{ccc|c} 0 & 3 & -6 & 8 \\ 1 & -2 & 3 & -1 \\ 5 & -7 & 9 & 0 \end{array}\right]$$

$$\Leftrightarrow \left[\begin{array}{ccc|c} & & & \\ & & & \\ 5 & -7 & 9 & 0 \end{array}\right]$$

$$\Leftrightarrow \left[\begin{array}{ccc|c} 1 & & & \\ 0 & & & \\ 0 & & & \end{array}\right]$$

$$\Longleftrightarrow \left[\begin{array}{ccc|c} 1 & -2 & 3 & -1 \\ 0 & & & \\ 0 & 0 & 0 & \end{array}\right]$$

The last row means

which is ________ to the linear system is ________.

<u>EXAMPLE</u> For which values of h is the following linear system consistent?

$$3x_1 - 9x_2 = 4$$
$$-2x_1 + 6x_2 = h$$

<u>Soln:</u>

$$\left[\begin{array}{cc|c} 3 & -9 & 4 \\ -2 & 6 & h \end{array}\right]$$

$$\Longleftrightarrow \left[\begin{array}{cc|c} 3 & -9 & 4 \\ 0 & & \end{array}\right]$$

which is consistent iff ______.

Section 1.2 ROW REDUCTION & ECHELON FORMS

Def$^{\underline{n}}$ A matrix is in echelon form (or ______ echelon form) if

① All nonzero rows are ___ ___ ___ ___ ___ ___ ___ ___ ___ ___ ___.
② Each leading entry (___ ___ ___ ___ ___ ___ ___ entry) of a row is in a column to the ___ ___ ___ ___ of the ___ ___ ___ ___ ___ ___ entry ___ ___ ___ it.
③ All entries in a column ___ ___ ___ a leading entry are ___ ___ ___ ___.

EXAMPLES

$$\begin{bmatrix} 1 & 2 & 3 & 1 & 3 & 6 \\ 0 & 0 & 4 & 2 & 4 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & -1 & 6 & 7 & 8 \\ 0 & 0 & 3 & 2 & 1 & 5 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

If in addition

④ The leading entry of each row is ___.
& ⑤ Each leading entry is the ___ ___ ___ ___ non-zero entry in its ___ ___ ___ ___ ___.

The matrix is said to be in ___ ___ ___ ___ ___ row echelon form.

Example

$$\begin{bmatrix} 1 & 4 & & 7 & 8 & & 9 \\ 0 & 0 & 1 & 5 & 6 & & 10 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$ is in

Defn Two matrices are called row equivalent if one can be obtained ________ by a ________ of ________ ____ operations.

NOTATION " ... " means "row equivalent".

$$\begin{bmatrix} 1 & 8 & -15 & 10 \\ 5 & 41 & -78 & 49 \\ 2 & 18 & -36 & 19 \end{bmatrix} \sim \begin{bmatrix} 1 & 8 & -15 & 10 \\ 0 & & & \\ 0 & & & \end{bmatrix}$$

THEOREM 1 (Uniqueness of Reduced Echelon Form)
Each matrix is ____ ________ to ____ and ______ ____ ______ ______ matrix.

TERMS:

pivot position — position of a ______ in an ________ matrix

pivot — a number (in a ______) used to create ______.

pivot column — a column that ____________.

<u>EXAMPLE</u> Row reduce to echelon form and locate pivot columns.

$$\begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & +5 & -9 & -7 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 4 & +5 & -9 & -7 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ & & & & \end{bmatrix}$$

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & & & & \\ 0 & & & & \\ 0 & & & & \end{bmatrix}$$

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 1 & 2 & -3 & -3 \\ 0 & 5 & 10 & -15 & -15 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 1 & 2 & -3 & -3 \\ 0 & 0 & & & \\ 0 & 0 & & & \end{bmatrix}$$

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 1 & 2 & -3 & -3 \\ 0 & 0 & 0 & & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (\qquad)$$

PIVOT COLUMNS

<u>EXAMPLE</u> Row reduce to echelon form and then to reduced echelon form:

$$\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix}$$

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & & & & & \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & & & & & \end{bmatrix}$$

$$(\qquad)$$

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 0 & - \\ 0 & 2 & -4 & 4 & 0 & \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -3 & 4 & -3 & 0 & -3 \\ 0 & 1 & & & 0 & \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & & & 0 & \\ 0 & 1 & & & 0 & \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

which is in ______________________.

SOLVING A LINEAR SYSTEM

1. Write the linear system as __________.
2. Reduce to __________________.
3. ______ ________ and solve using ______ variables if necessary.

EXAMPLE Solve the linear system whose augmented matrix has been reduced to

$$\begin{bmatrix} 1 & 6 & 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & -8 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{bmatrix}$$

BASIC VARIABLES

FREE VARIABLES

$$\begin{aligned} x_1 + 6x_2 \quad + 3x_4 &= 0 \\ x_3 - 8x_4 &= 5 \\ x_5 &= 7 \end{aligned}$$

$x_1 =$

$x_3 =$

$x_5 =$

and

$x_5 =$
$x_4 =$
$x_3 =$
$x_2 =$
$x_1 =$

The general solution is given by

$(x_1, x_2, x_3, x_4, x_5) =$

The Solution Set

$= \{ (\quad , \quad , \quad , \quad , \quad) : \quad \}.$

$x_5 =$

$x_4 =$

$x_3 =$

$x_2 =$

$x_1 =$

The general solution is given by

$(x_1, x_2, x_3, x_4, x_5) =$

The Solution Set

$= \{ (\quad , \quad , \quad , \quad , \quad) : \qquad \}.$

(p. 18)

EXISTENCE & UNIQUENESS of SOLUTIONS

Example Determine whether the system

$$
\begin{aligned}
3x_2 - 6x_3 + 6x_4 + 4x_5 &= -5 \\
3x_1 - 7x_2 + 8x_3 - 5x_4 + 8x_5 &= 9 \\
3x_1 - 9x_2 + 12x_3 - 7x_4 + 6x_5 &= 15
\end{aligned}
$$

is consistent. If it is, does it have infinitely many solutions?

In an earlier problem, we reduced the augmented matrix to

$$
\begin{bmatrix}
3 & -9 & 12 & -9 & 6 & 15 \\
0 & 2 & -4 & 4 & 2 & -6 \\
0 & 0 & 0 & 0 & &
\end{bmatrix}
$$

which is in ___________.

No row has the form

$$
\begin{bmatrix}
\quad & \quad & \quad
\end{bmatrix}
$$

so the system is ________.

Since __________ are free variables the system has ______________.

Example Determine whether the system

$$3x_1 + 4x_2 = -3$$
$$2x_1 + 5x_2 = 5$$
$$-2x_1 - 3x_2 = 1$$

is consistent. If it is, determine the number of solutions.

$$\left[\begin{array}{cc|c} 3 & 4 & -3 \\ 2 & 5 & 5 \\ -2 & -3 & 1 \end{array}\right] \sim \left[\begin{array}{cc|c} 3 & 4 & -3 \\ 6 & & \\ -6 & & \end{array}\right]$$

$$\sim \left[\begin{array}{cc|c} 3 & 4 & -3 \\ 0 & & \\ 0 & & \end{array}\right] \sim \left[\begin{array}{cc|c} 3 & 4 & -3 \\ 0 & & \\ 0 & & \end{array}\right]$$

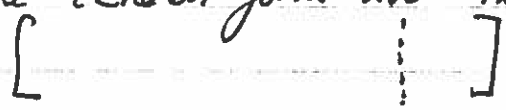
which is ________.
Since ...., the solution is ________.

(p. 20)

THEOREM 2 (Existence & Uniqueness Theorem)

(1) A linear system is consistent iff the ____ column is ________.

ie the echelon form does not contain a row of the form

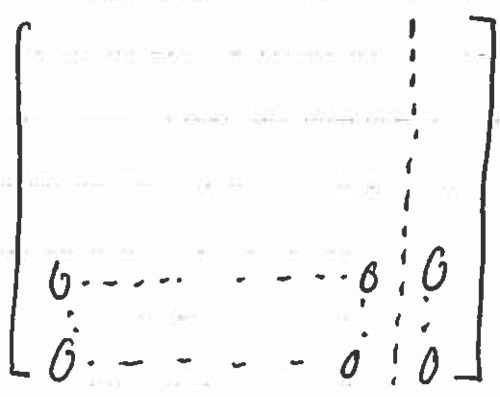


where

(2) If a linear system is consistent, then the solution set contains either

(i) a unique solution ie when there are ___ free variables

ie when the number of pivots ____ the number of columns in the ______ matrix.



(ii) infinitely many solutions when there is ________ free variable ie number of pivots ________

NOTE: A consistent system with more ______

Theorem ______ must have ______ many solutions

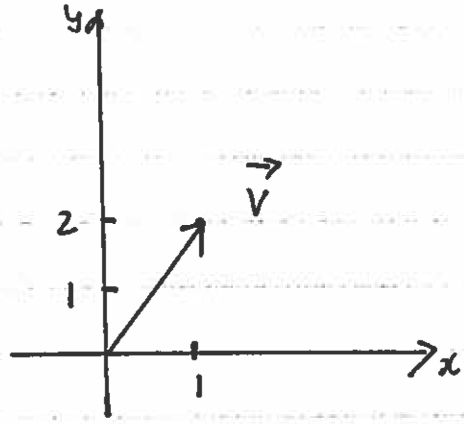
NOTE: In a consistent system

\# of pivots $\leq$ \# of rows = ____________.
because pivots are in ____________.

\# of pivots $\leq$ \# of columns of the coefficent matrix
= ____________________
because pivots are in ____________________.
and there ____________________.

\# of free variables =

Section 1.3 VECTOR EQUATIONS



Notations for $\vec{v}$

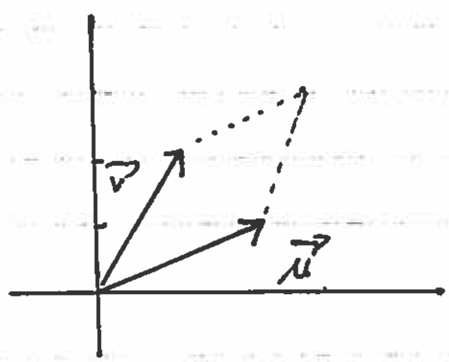
①

②

③

A general vector in the real plane $\mathbb{R}^2$ has the form where

Vector Addition



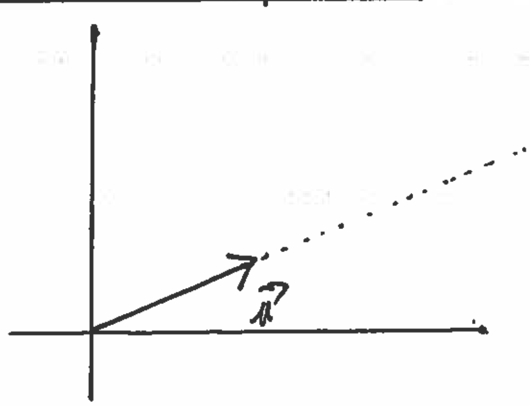
If $\vec{u} = \begin{bmatrix} a \\ b \end{bmatrix}$, $\vec{v} = \begin{bmatrix} c \\ d \end{bmatrix}$, then

$$\vec{u} + \vec{v} = \begin{bmatrix} \ \\ \ \end{bmatrix}.$$

The zero vector $\vec{0} = \begin{bmatrix} \ \\ \ \end{bmatrix}$

$\vec{u} + (-\vec{u}) = \dots$ where $-\vec{u} = \begin{bmatrix} \ \\ \ \end{bmatrix}$.

Scalar Multiplication

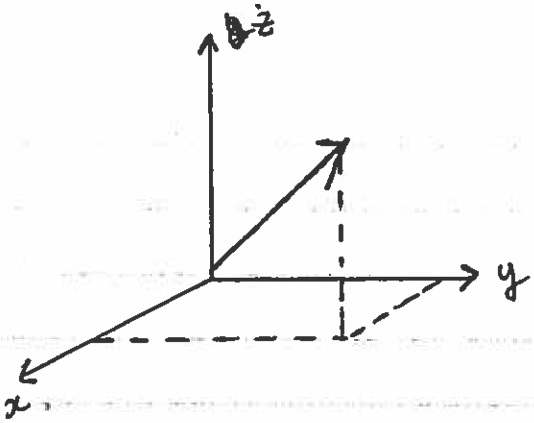


Let $\alpha \in \mathbb{R}$, $\vec{u} = \begin{bmatrix} a \\ b \end{bmatrix} \in \mathbb{R}^2$. Then

$$\alpha \vec{u} = \alpha \begin{bmatrix} \ \\ \ \end{bmatrix} = \begin{bmatrix} \ \\ \ \end{bmatrix}$$

(p.23)

$\mathbb{R}^3$



$\vec{v} = \begin{bmatrix} \\ \\ \\ \end{bmatrix}$

Vectors in $\mathbb{R}^n$

Let $n \geq 1$ be an integer. Then

$$\mathbb{R}^n = \left\{ \begin{bmatrix} \\ \\ \\ \end{bmatrix} : \qquad \right\}$$

For example, $\begin{bmatrix} \\ \\ \\ \end{bmatrix} \in \mathbb{R}^4$.

Vector addition & Scalar multiplication

Let $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}, \vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \in \mathbb{R}^n$ & $\alpha \in \mathbb{R}$

Then $\vec{u} + \vec{v} = \begin{bmatrix} \\ \\ \\ \end{bmatrix}$, $\alpha \vec{u} = \begin{bmatrix} \\ \\ \\ \end{bmatrix}$

The zero vector $\vec{0} = \begin{bmatrix} \\ \\ \\ \end{bmatrix}$, $-\vec{u} = \begin{bmatrix} \\ \\ \\ \end{bmatrix}$.

Algebraic Properties of Vectors in $\mathbb{R}^n$

Let $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^n$ & $\alpha, \beta \in \mathbb{R}$.
Then

(1) $\vec{u} + \vec{v} =$

(2) $(\vec{u} + \vec{v}) + \vec{w} =$

(3) $\vec{u} + \vec{0} =$

(4) $\vec{u} + (-\vec{u}) =$ where $-\vec{u} =$

(5) $\alpha(\vec{u} + \vec{v}) =$

(6) $(\alpha + \beta)\vec{u} =$

(7) $\alpha(\beta \vec{u}) =$

(8) $1\vec{u} =$

LINEAR COMBINATIONS

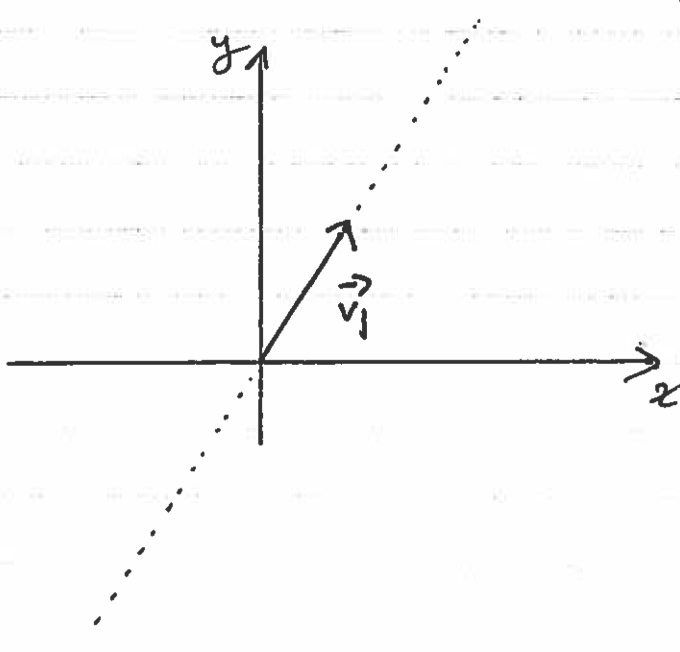
Def: Let $\vec{v_1}, \vec{v_2}, \ldots, \vec{v_m} \in \mathbb{R}^n$. A vector $\vec{u} \in \mathbb{R}^n$ is a linear combination of the vectors $\vec{v_1}, \vec{v_2}, \ldots, \vec{v_m}$ if

EXAMPLE Let $\vec{v_1} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \in \mathbb{R}^2$ & $\vec{u} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$.

Then $\vec{u} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$ is ____________

since

Any vector in ____________ is a linear combination of $\vec{v_1}$.



An Example in $\mathbb{R}^3$

Let $\vec{v_1} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\vec{v_2} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ (p. 26)

Determine if

(1) $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ is a linear combination of $\vec{v_1}, \vec{v_2}$.

(2) $\begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$ is a linear combination of $\vec{v_1}, \vec{v_2}$.

(3) Determine the set of vectors that are linear combinations of $\vec{v_1}, \vec{v_2}$.

(1) $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ is a linear combination of $\vec{v_1}, \vec{v_2}$ if there are constants α_1, α_2 such that

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} =$$

$$\Leftrightarrow \left\{ \begin{array}{l} \\ \\ \\ \end{array} \right.$$

which is $\_\_\_\_\_\_$ so $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ is

(2) Consider the vector equation

$$\begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} =$$

$\Longleftrightarrow \left\{ \begin{array}{l} \\ \\ \end{array} \right.$

This system is ______ with __________

so $\begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$ is

In fact,

$$\begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} =$$

(3) Let $\vec{u} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3$

$\vec{u}$ is a linear combination of $\vec{v_1}, \vec{v_2}$ if

$\Longleftrightarrow$

$\Longleftrightarrow \begin{bmatrix} & \\ & \end{bmatrix} = \begin{bmatrix} \\ \\ \end{bmatrix}$

$\Longleftrightarrow \left[\begin{array}{c|c} & \\ & \end{array}\right] \longleftrightarrow \left[\begin{array}{c|c} & \\ & \end{array}\right]$

$$\longleftrightarrow \left[\quad \vdots \quad\right] \longleftrightarrow \left[\quad \vdots \quad\right]$$

which is consistent iff

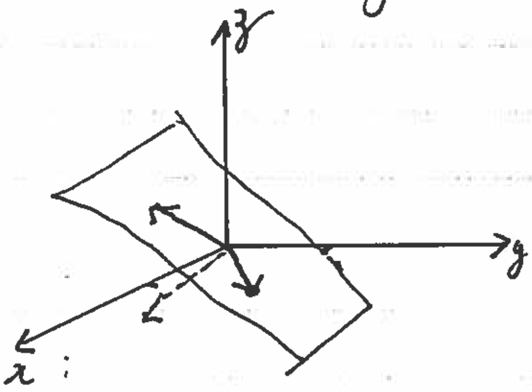
So $\vec{u} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is a linear combination of $\vec{v_1}, \vec{v_2}$

iff

The set of vectors that are linear combinations of $\vec{v_1}, \vec{v_2}$

$$= \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : \qquad \qquad \right\}$$

<u>NOTE</u> ① $x - y - z = 0$ is the equation of the ______ with ____ vector ________ and which goes through the point (, ,).

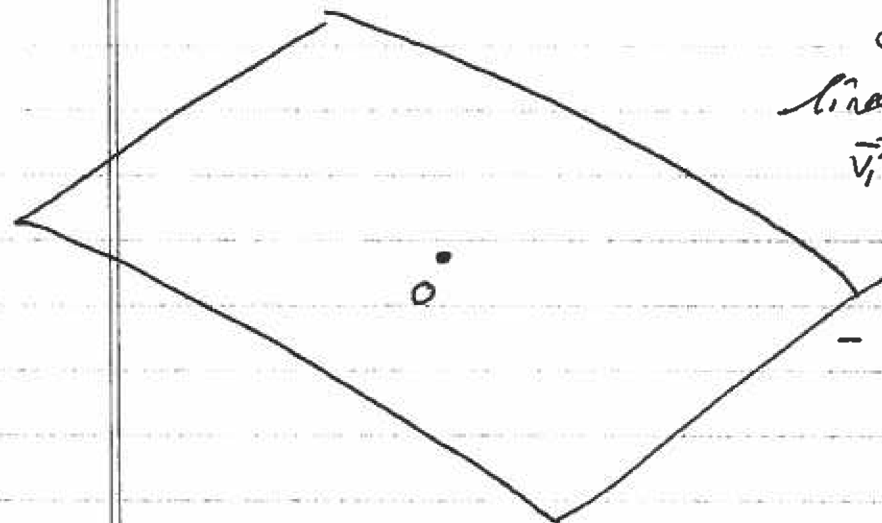


$$\vec{v_1} \times \vec{v_2} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix}$$

$$= \vec{i} \begin{vmatrix} & \end{vmatrix} - \vec{j} \begin{vmatrix} & \end{vmatrix} + \vec{k} \begin{vmatrix} & \end{vmatrix}$$

$$=$$

(2) Let $\vec{v_1}, \vec{v_2}$ be two non-zero vectors in $\mathbb{R}^3$ that are not linear combinations of each other (ie do not ____________).



Then the set of linear combinations of $\vec{v_1}, \vec{v_2}$ form a __________ __________.

(3) In general, $\vec{u}$ is a linear combination of vectors $\vec{v_1}, \vec{v_2}, \ldots, \vec{v_k}$ in $\mathbb{R}^n$ if

=

has a ______________.
This is equivalent to the linear system

$$\begin{bmatrix} \quad \end{bmatrix}$$

being ______.

(p.30)

Example

Let $\vec{v_1} = \begin{bmatrix} 1 \\ 6 \\ 10 \\ 14 \end{bmatrix}$, $\vec{v_2} = \begin{bmatrix} 1 \\ 5 \\ 12 \\ 19 \end{bmatrix}$, $\vec{v_3} = \begin{bmatrix} 1 \\ 10 \\ 1 \\ -8 \end{bmatrix}$.

Is $\vec{u} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ a linear combination of $\vec{v_1}, \vec{v_2}, \vec{v_3}$?

$\vec{u}$ is a linear combination of $\vec{v_1}, \vec{v_2}, \vec{v_3}$
iff there are __________ such that

____________ = ____________

$$\Longleftrightarrow \quad \begin{bmatrix} 1 \\ 6 \\ 10 \\ 14 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 5 \\ 12 \\ 19 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 10 \\ 1 \\ -8 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\Leftrightarrow \quad \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 6 & 5 & 10 & 0 \\ 10 & 12 & 1 & 0 \\ 14 & 19 & -8 & 1 \end{array}\right]$$

$$\Leftrightarrow \quad \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & & & \\ 0 & & & \\ 0 & & & \end{array}\right]$$

$$\Leftrightarrow \left[\begin{array}{ccc:c} 1 & 1 & 1 & 1 \\ 0 & 1 & -4 & 6 \\ 0 & 0 & & \\ 0 & 0 & & \end{array}\right]$$

$$\Leftrightarrow \left[\begin{array}{ccc:c} 1 & 1 & 1 & 1 \\ 0 & 1 & -4 & 6 \\ 0 & 0 & & \\ 0 & 0 & & \end{array}\right]$$

which is ________ so that
$\vec{u}$ ____________ of $\vec{v_1}, \vec{v_2}, \vec{v_3}$.

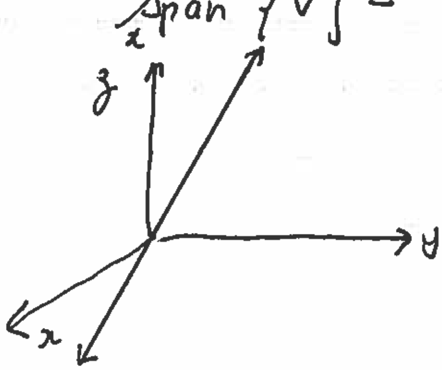
SPAN

Def$\underline{n}$ Let $\vec{v_1}, \vec{v_2}, \ldots, \vec{v_k} \in \mathbb{R}^n$.
The set of all ____________ of $\vec{v_1}, \vec{v_2}, \vec{v_3}, \ldots, \vec{v_k}$ is called the ____________ ________ by $\vec{v_1}, \vec{v_2}, \ldots, \vec{v_k}$ and is denoted by ____ $\{\vec{v_1}, \vec{v_2}, \ldots, \vec{v_k}\}$.

NOTE

① If $\vec{v} \neq \vec{0}$ and $\vec{v} \in \mathbb{R}^3$ then

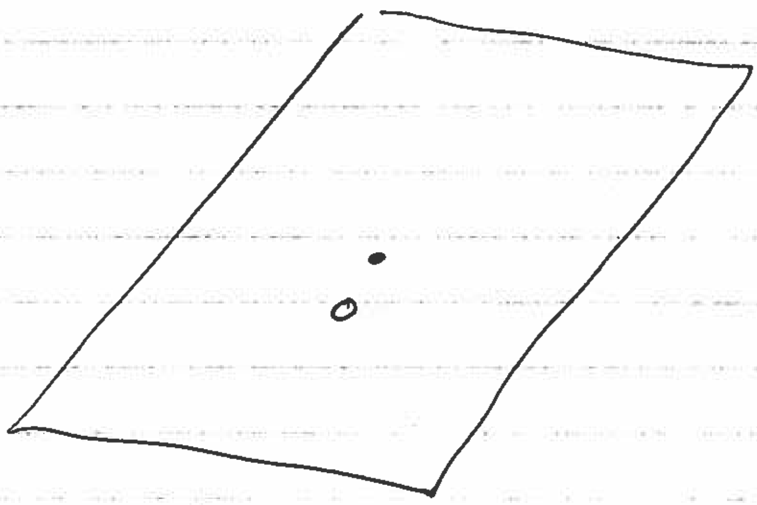
$$\text{Span}\{\vec{v}\} = \{ \qquad\qquad \}$$



② Let $\vec{v_1}, \vec{v_2} \in \mathbb{R}^3$, $\vec{v_1}, \vec{v_2} \neq \vec{0}$ and $\vec{v_1}, \vec{v_2}$ not scalar multiples of each other.
Then

$$\text{Span}\{\vec{v_1}, \vec{v_2}\} = \{ \qquad\qquad\qquad\qquad \}$$

is a _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ .



EXAMPLE
Let

$$\vec{v_1} = \begin{bmatrix} 1 \\ 6 \\ 10 \\ 14 \end{bmatrix}, \quad \vec{v_2} = \begin{bmatrix} 1 \\ 5 \\ 12 \\ 19 \end{bmatrix}, \quad \vec{v_3} = \begin{bmatrix} 1 \\ 10 \\ 1 \\ -8 \end{bmatrix} \in \mathbb{R}^4$$

Determine $\text{Span}\{\vec{v_1}, \vec{v_2}, \vec{v_3}\}$.

Let $\vec{u} = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \in \mathbb{R}^4$.

$\vec{u} \in Span\{\vec{v_1}, \vec{v_2}, \vec{v_3}\}$ iff

$\vec{u} =$

$\Leftrightarrow$ The linear system

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & \\ 6 & 5 & 10 & \\ 10 & 12 & 1 & \\ 14 & 19 & -8 & \end{array}\right]$$

is ________.

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & x \\ 0 & & & \\ 0 & & & \\ 0 & & & \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & x \\ 0 & 1 & -4 & 6x-y \\ 0 & 0 & & \\ 0 & 0 & & \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & x \\ 0 & 1 & -4 & 6x-y \\ 0 & 0 & -1 & z+2y-22x \\ 0 & 0 & & \end{array}\right]$$

which is consistent iff ________.

Hence

$$\text{Span}\{\vec{v_1}, \vec{v_2}, \vec{v_3}\} = \left\{ \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \in \mathbb{R}^4 : \qquad \right\}$$

<u>Def^n</u> Let $S = \{\vec{v_1}, \vec{v_2}, \dots, \vec{v_m}\} \subset \mathbb{R}^n$.
We say the set S <u>spans</u> $\mathbb{R}^n$ if

_ _ _ _ _ _ _ _ _ _

<u>ie</u> _ _ _ _ vector in $\mathbb{R}^n$ can be _ _ _ _ _
as a _ _ _ _ _ _ _ _ _ _ _ _ of vectors in _ _ _.

<u>Example</u> In the example above the set $\{\vec{v_1}, \vec{v_2}, \vec{v_3}\}$

_ _ _ _ _ _ _ _ _ _ _ _ .

However the vectors

$$\vec{e_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{e_2} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{e_3} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad \vec{e_4} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

_ _ _ _ _ _ _ _ _ . Any vector $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \in \mathbb{R}^4$

can be written as

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Section 1.4 The Matrix Equation $A\vec{x} = \vec{b}$

Definition of $A\vec{x}$

Let $m, n \geq 1$, Suppose

$$\vec{a_1}, \vec{a_2}, \ldots, \vec{a_n} \in \mathbb{R}^m$$

and

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n.$$

Let A be the __ x __ matrix whose columns are the vectors $\vec{a_1}, \vec{a_2}, \ldots, \vec{a_n}$;

$$A = \begin{bmatrix} \quad \quad \quad \end{bmatrix}.$$

Then

$$A\vec{x} := \ldots\ldots\ldots\ldots\ldots$$

NOTE ① $A\vec{x} \in$ ____ if A is __ x __ and $\vec{x} \in \mathbb{R}^{--}$.

② $A\vec{x}$ is a ____________.

Example Let

$$\vec{a_1} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \vec{a_2} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}, \vec{a_3} = \begin{bmatrix} 3 \\ 4 \end{bmatrix} \in \mathbb{R}^2.$$

Let $A = [\vec{a_1} \mid \vec{a_2} \mid \vec{a_3}] = \begin{bmatrix} \quad \quad \quad \end{bmatrix}.$

Let $\vec{x} = \begin{bmatrix} \quad \end{bmatrix} \in$ ____.

Then

$$A\vec{x} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$=$$

For example,

$$A \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} =$$

LINEAR EQUATION IN ONE VARIABLE (High School math)

Let $a, b \in \mathbb{R}$, $x \in \mathbb{R}$.

Consider the equation

$$ax = b \qquad (*).$$

① If $a \neq 0$ then ______________.

② If $a = b = 0$ then ______________.

③ If $a = 0$ and $b \neq 0$ then ______________.

The matrix equation $A\vec{x}=\vec{b}$

Suppose $\vec{a_1}, \vec{a_2}, \dots, \vec{a_n}, \vec{b} \in \mathbb{R}^m$.

Then

$\vec{b}$ is a linear combination of $\vec{a_1}, \vec{a_2}, \dots, \vec{a_n}$

iff $\vec{b} =$

for some

$\Longleftrightarrow$ $\vec{b} =$

where

$A = [\qquad\qquad]$.

NOTE $A\vec{x} = \vec{b}$ for some $\vec{x} \in \mathbb{R}^n$

iff $\vec{b} \in$ ....

Example Let

$$A = \begin{bmatrix} 6 & 5 & 10 \\ -3 & -2 & -7 \\ -5 & -3 & -13 \end{bmatrix}$$

(i) Is the matrix equation $A\vec{x} = \vec{b}$ consistent for all $\vec{b} \in \mathbb{R}^3$?

(ii) Rewrite your answer to (i) in terms of Span.

(p.38)

(i) Let $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$.

$$A\vec{x} = \vec{b}$$

$$\Leftrightarrow \quad \vec{b} =$$

$$\Leftrightarrow \left[\begin{array}{ccc:c} 6 & 5 & 10 & \\ -3 & -2 & -7 & \\ -5 & -3 & -13 & \end{array}\right]$$

$$\Leftrightarrow \left[\begin{array}{ccc:c} & & & \\ -3 & -2 & -7 & b_2 \\ -5 & -3 & -13 & b_3 \end{array}\right]$$

$$\Leftrightarrow \left[\begin{array}{ccc:c} 1 & 2 & -3 & b_1 + b_3 \\ 0 & & & \\ 0 & & & \end{array}\right]$$

$$\Leftrightarrow \left[\begin{array}{ccc:c} 1 & 2 & -3 & b_1 + b_3 \\ 0 & 1 & -4 & \\ 0 & & & \end{array}\right]$$

$$\Leftrightarrow \left[\begin{array}{ccc:c} 1 & 2 & -3 & b_1 + b_3 \\ 0 & 1 & -4 & \\ 0 & 0 & & \end{array}\right]$$

This system is consistent

iff

ie

So

$A\vec{x} = \vec{b}$ is ....

$A\vec{x} = \vec{b}$ is consistent iff ....

(ii) .... can be written as a linear combination of the vectors

$$\vec{a}_1 = \begin{bmatrix} 6 \\ -3 \\ -5 \end{bmatrix}, \quad \vec{a}_2 = \begin{bmatrix} 5 \\ -2 \\ -3 \end{bmatrix}, \quad \vec{a}_3 = \begin{bmatrix} 10 \\ -7 \\ -13 \end{bmatrix};$$

ie The columns of A _ _ _ _ _ _ _ _ _ _ _.

Theorem Let A be an $m \times n$ matrix. The following statements are equivalent:

(i) For every $\vec{b} \in$ ___, $A\vec{x} = \vec{b}$ has ________ $\vec{x} \in$ ___.

(ii) The columns of A __________

(iii) A has a pivot position ________.

Example

Let $\vec{v_1} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$, $\vec{v_2} = \begin{bmatrix} 4 \\ 1 \\ 14 \end{bmatrix}$, $\vec{v_3} = \begin{bmatrix} -1 \\ 5 \\ 29 \end{bmatrix}$, $\vec{v_4} = \begin{bmatrix} 0 \\ 1 \\ 11 \end{bmatrix}$.

Show that the vectors $\vec{v_1}, \vec{v_2}, \vec{v_3}, \vec{v_4}$ span $\mathbb{R}^3$.

Let

$$A = \begin{bmatrix} 1 & 4 & -1 & 0 \\ 0 & 1 & 5 & 1 \\ 2 & 14 & 29 & 11 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 4 & -1 & 0 \\ 0 & 1 & 5 & 1 \\ 0 & & & \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 4 & & \\ 0 & 1 & & \\ 0 & 0 & & \end{bmatrix}$$

There is a pivot position ________ so

$\vec{v_1}, \vec{v_2}, \vec{v_3}, \vec{v_4}$ ____________.

Explanation: The vectors $\vec{v_1}, \vec{v_2}, \vec{v_3}, \vec{v_4}$ span $\mathbb{R}^3$

if ____________

is a __________ of $\vec{v_1}, \vec{v_2}, \vec{v_3}, \vec{v_4}$

ie $x_1 \vec{v_1} + x_2 \vec{v_2} + x_3 \vec{v_3} + x_4 \vec{v_4} = \vec{b}$

is ________________

$\iff \quad A\vec{x} = \vec{b}$ is - - - - - - - - - - -

where $A = \begin{bmatrix} & & \end{bmatrix}$

$$\iff \left[\begin{array}{cccc|c} 1 & 4 & -1 & 0 & \\ 0 & 1 & 5 & 1 & \\ 2 & 14 & 29 & 11 & \end{array}\right]$$

is - - - - - - - - - - - -

Now $\left[\begin{array}{cccc|c} 1 & 4 & -1 & 0 & \\ 0 & 1 & 5 & 1 & \\ 2 & 14 & 29 & 11 & \end{array}\right]$

$$\sim \left[\begin{array}{cccc|c} 1 & 4 & -1 & 0 & \\ 0 & 1 & 5 & 1 & \\ 0 & 0 & 1 & 5 & \end{array}\right]$$

which is - - - - - - - - .

<u>Theorem</u> Let A be an $m \times n$ matrix, $\vec{u}, \vec{v} \in \mathbb{R}^n$ and $c \in \mathbb{R}$. Then

(i) $A(\vec{u} + \vec{v}) =$

(ii) $A(c\vec{u}) =$

(p.42)

Proof: Let $A = \left[\, \vec{a}_1 \mid \vec{a}_2 \mid \cdots \mid \vec{a}_n \,\right]$,

$$\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}.$$

Then

$$\vec{u} + \vec{v} = \begin{bmatrix} \quad \\ \quad \\ \quad \\ \quad \end{bmatrix}, \text{ and } c\vec{u} = \begin{bmatrix} \quad \\ \quad \\ \quad \\ \quad \end{bmatrix}.$$

So

$$A(\vec{u} + \vec{v}) =$$

$$=$$

$$=$$

$$A(c\vec{u}) =$$

$$=$$

$$=$$

$\square$

Section 1.5 Solution Sets of Linear Systems

Homogeneous Linear Systems

Def$^{\underline{n}}$ A system of linear equations is homogeneous if it has the form

$$A\vec{x} = \ldots$$

where

A is ______, $\vec{x} =$, $\vec{0} =$.

Example The system

$$x_1 + 2x_2 - x_3 = 0$$
$$4x_1 - x_2 + 3x_3 = 0$$

is a homogeneous linear system. It can be written as

$$\begin{bmatrix} & & \\ & & \end{bmatrix}\begin{bmatrix} \\ \\ \\ \end{bmatrix} = \begin{bmatrix} \\ \\ \end{bmatrix}.$$

NOTE

(i) This system has at least one solution since $(x_1, x_2, x_3) = (\ \ ,\ \ ,\ \ )$ is a solution.

(ii) In general $\vec{x} = \ldots$ is a solution to

$$A\vec{x} = \ldots .$$

The solution $\vec{x} = \ldots$ is called the _____ solution.

(iii) The homogeneous system $A\vec{x} = \vec{0}$ has a non-trivial solution iff the system has __ __ __ __ __ __ __ __ __ .

Example

(i) Determine if the linear system

$$\begin{aligned} -x_1 + 3x_2 + 11x_3 &= 0 \\ -5x_1 + 10x_2 + 41x_3 &= 0 \\ 8x_1 - 4x_2 - 32x_3 &= 0 \end{aligned}$$

has a non-trivial solution

(ii) Determine all solutions. Write the general solution in parametric vector form.

$$\left[\begin{array}{ccc|c} -1 & 3 & 11 & 0 \\ -5 & 10 & 41 & 0 \\ 8 & -4 & -32 & 0 \end{array}\right]$$

$$\Leftrightarrow \left[\begin{array}{ccc|c} -1 & 3 & 11 & 0 \\ -5 & 10 & 41 & 0 \\ & & & \end{array}\right]$$

$$\Leftrightarrow \left[\begin{array}{ccc|c} -1 & 3 & 11 & 0 \\ 0 & & & \\ 0 & & & \end{array}\right]$$

$$\Leftrightarrow \left[\begin{array}{ccc|c} -1 & & & \\ 0 & & & \\ 0 & 0 & & \end{array}\right]$$

which is in __ __ __ __ __ .

The variable _________.
So the linear system has _____________.

(ii) Continuing:

$$\cdots \leftrightarrow \left[\begin{array}{ccc|c} -1 & & & 0 \\ 0 & & & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\leftrightarrow \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

___ is a free variable. Let ___ $= t$.

$x_2 =$

$x_1 =$

So the general solution is given by

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \\ \\ \\ \end{bmatrix} = t \begin{bmatrix} \\ \\ \\ \end{bmatrix}$$

where $t \in$ ____.
This is the _____________ of the solution.

Example Let

$$A = \begin{bmatrix} 1 & 2 & -2 & -3 \\ 3 & 6 & -5 & -11 \\ -5 & -10 & 12 & 11 \\ 5 & 10 & -14 & -7 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 8 \\ 31 \\ -26 \\ 12 \end{bmatrix}$$

Show that the linear system

$$A\vec{x} = \vec{b}$$

is consistent and write the general solution in parametric vector form.

$$A\vec{x} = \vec{b}$$

$$\Leftrightarrow \left[\begin{array}{cccc|c} 1 & 2 & -2 & -3 & 8 \\ 3 & 6 & -5 & -11 & 31 \\ -5 & -10 & 12 & 11 & -26 \\ 5 & 10 & -14 & -7 & 12 \end{array}\right]$$

$$\Leftrightarrow \left[\begin{array}{cccc|c} 1 & 2 & -2 & -3 & 8 \\ 0 & & & & \\ 0 & & & & \\ 0 & & & & \end{array}\right]$$

$$\Leftrightarrow \left[\begin{array}{cccc|c} 1 & & & & 8 \\ 0 & & & & \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\Leftrightarrow \left[\begin{array}{cccc|c} 1 & & & & \\ 0 & 0 & 1 & & \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

______ are free variables.

Let $\quad = t, \quad = s.$ Then

$=$

$=$

$=$

So the general solution is

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \\ \\ \\ \\ \end{bmatrix}$$

$$= \begin{bmatrix} \\ \\ \\ \\ \end{bmatrix} \quad \begin{bmatrix} \\ \\ \\ \\ \end{bmatrix} \quad \begin{bmatrix} \\ \\ \\ \\ \end{bmatrix}$$

where

NOTE Letting $t = s = 0$ we see that one solution to $A\vec{x} = \vec{b}$ is

$$\vec{x} = \begin{bmatrix} \\ \\ \\ \\ \end{bmatrix}.$$

<u>Example</u> For the same 4×4 matrix A, solve $A\vec{x} = \vec{0}$. p.48

$$A\vec{x} = \vec{0} \iff \left[\begin{array}{cccc|c} 1 & 2 & -2 & -3 & 0 \\ 3 & 6 & -5 & -11 & 0 \\ -5 & -10 & 12 & 11 & 0 \\ 5 & 10 & -14 & -7 & 0 \end{array}\right]$$

$$\vdots$$

$$\iff \left[\begin{array}{cccc|c} 1 & & & & 0 \\ 0 & & & & 0 \\ 0 & & & & 0 \\ 0 & & & & 0 \end{array}\right]$$

The general solution is given by

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \\ \\ \\ \\ \end{bmatrix}$$

$$= \begin{bmatrix} \\ \\ \\ \\ \end{bmatrix} \qquad \begin{bmatrix} \\ \\ \\ \\ \end{bmatrix}$$

where.

<u>NOTE</u> The solution above appears as part of the solution to
$$A\vec{x} = \vec{b}.$$
In fact the general solution to
$$A\vec{x} = \vec{b}$$
can be written as
$$\vec{x} = \qquad + $$

where

<u>Theorem</u> Let A be an $m \times n$ matrix, and suppose $\vec{b} \in \mathbb{R}^m$. Suppose

$$(*) \qquad A\vec{x} = \vec{b}$$

is ________ and $\vec{x} =$ is a ________.
Then the general solution of $(*)$ can be written as

$$\vec{x} =$$

where

<u>PROOF:</u> Suppose $\vec{p}$ and $\vec{w}$ are solutions to $(*)$.
Then
$$A\vec{p} =$$
$$A\vec{w} =$$
$$A(\vec{w} - \vec{p}) =$$

So $\vec{v}_h =$ is a solution to $A\vec{x} =$ ___,
and

$\vec{w} =$

where $\vec{v}_h$ is a ___________.

Conversely, let

$$\vec{w} = \vec{p} + \vec{v}_h$$

where $\vec{x} = \vec{p}$ is a ______ and
$\vec{x} = \vec{v}_h$ is a ______. Then

$$A\vec{w} =$$

$$=$$

and $\vec{x} = \vec{w}$ is a ________ $\square$

Section 1.7 Linear Independence

Question: Suppose we are given a set of vectors

$$S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p\}.$$

Is there a linear relation between $\vec{v}_1, \ldots, \vec{v}_p$?
This means: Is one of the vectors a linear combination of the others? If this is the case we say that the vectors are ____________________.

Definition:
Suppose $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p\} \subset \mathbb{R}^n$.
The vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p$ are linearly independent (or the set S is ____________) if the vector equation

$$=$$

has ____________________________
Otherwise we say the vectors (or the set S) are __________________. In this case there are ____________________________.
such that

$$=$$

Example The vectors $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} 3 \\ 5 \\ 7 \end{bmatrix}$

are linearly __________ since

$$\vec{v}_3 =$$

and

$$= .$$

Alternatively, consider the vector equation

$=$

$\iff$

$\sim$

$\sim$

<u>Example</u> Let

$$A = \begin{bmatrix} 1 & 6 & 8 \\ -1 & -5 & -9 \\ 1 & 6 & 9 \\ 1 & -2 & -11 \end{bmatrix}$$

Determine if the columns of A are linearly independent.

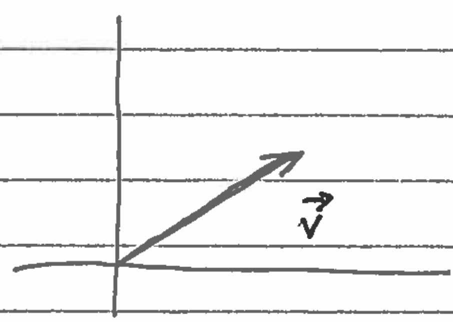
Sets of vectors and Linear Independence

① $\{\vec{0}\}$ is linearly _______ (_____),

since

② $\{\vec{v}\}$ is linearly _______ if $\vec{v} \neq \vec{0}$.

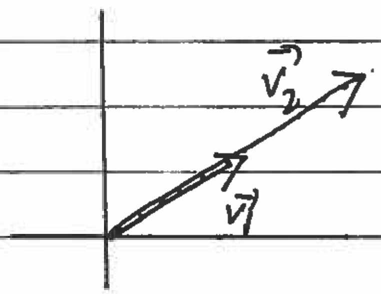
PROOF: Suppose $x_1 \vec{v} = \vec{0}$ where $x_1 \in \mathbb{R}$.

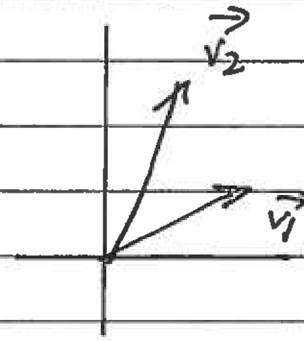


③ Let $\{\vec{v_1}, \vec{v_2}\} \subset \mathbb{R}^n$.
Suppose $\{\vec{v_1}, \vec{v_2}\}$ is linearly dependent.
Then

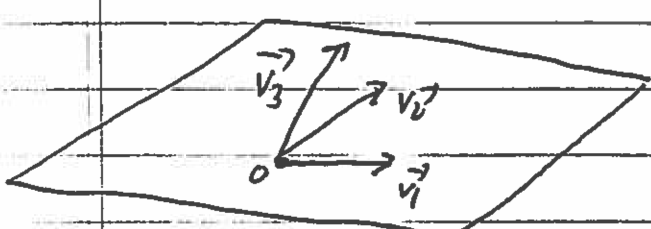
Hence if $\{\vec{v}_1, \vec{v}_2\}$ is linearly dependent then either

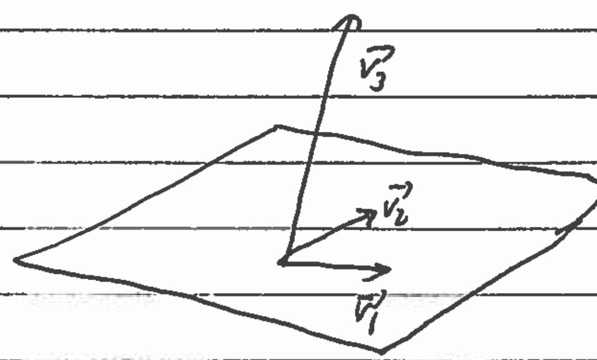
Therefore if $\vec{v}_1, \vec{v}_2$ are _ _ _ _ _ _ _ _ _ _ .
then $\{\vec{v}_1, \vec{v}_2\}$ is linearly independent.





<u>Theorem</u>
Let $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p\} \subset \mathbb{R}^n$ be a set of two or more vectors. Then S is linearly dependent iff





<u>Theorem</u> Any set containing the zero vector $\vec{0}$ is linearly _ _ _ _ _ _ .

NOTE

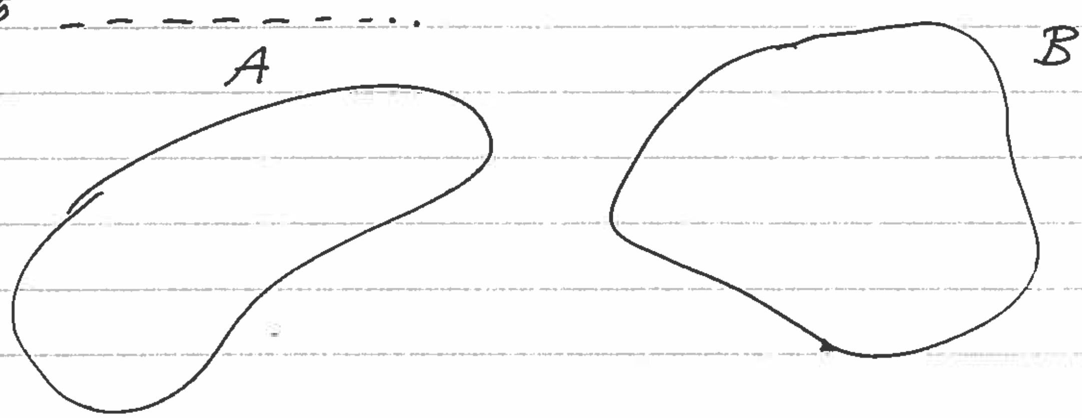
① Suppose $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p\} \subset \mathbb{R}^n$ is linearly dependent (where $p \geq 2$).
Then
$$x_1 \vec{v}_1 + x_2 \vec{v}_2 + \cdots + x_p \vec{v}_p = \vec{0}$$
for ______

② Consider $S = \{\vec{0}, \vec{v}_2, \ldots, \vec{v}_p\} \subset \mathbb{R}^n$. Then
$$(\ )\vec{0} + (\ )\vec{v}_2 + \cdots + (\ )\vec{v}_p = \vec{0}$$
and so S is linearly ________.

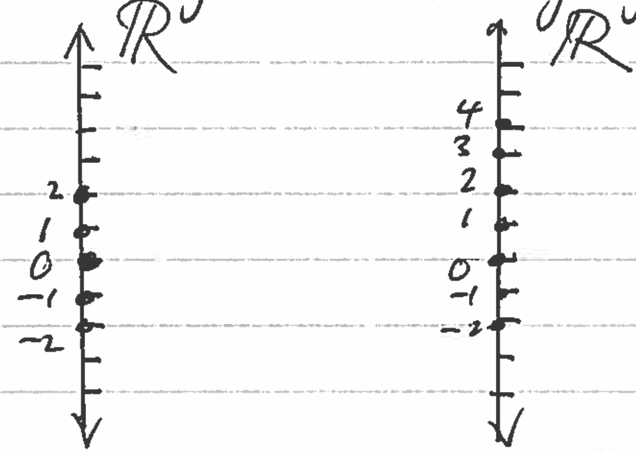
③ Theorem: Let A be an $m \times n$ matrix.
The columns of A are linearly independent iff

Section 1.8 Introduction to Linear Transformations

Function Notation Let A, B be sets. Then a function $f: A \to B$ is an _________________ to _____________.



Example Let $f: \mathbb{R} \to \mathbb{R}$ by $f(x) = x^2$.



MATRIX TRANSFORMATION

Let A be an $m \times n$ matrix. The transformation

$$T: \ldots \longrightarrow \ldots$$ defined by

$$T(\vec{x}) = A\vec{x}$$

is called a <u>matrix transformation</u>.

(p. 58)

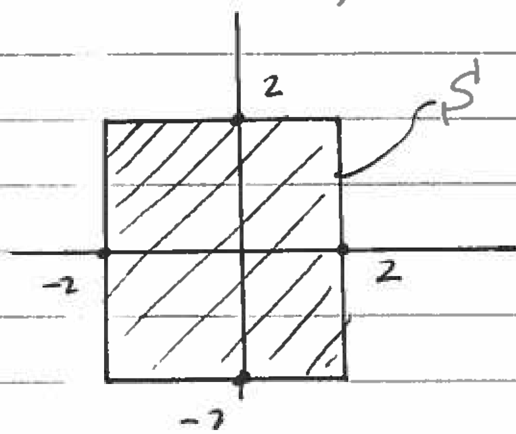
Example Define $T: \mathbb{R}^2 \to \mathbb{R}^2$ by $T(\vec{x}) = A\vec{x}$ where

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

(i) Find $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right)$.

(ii) Find $T\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right)$.

(iii) Find the image of the square S under the transformation T.



(i)

(ii)

(iii) We find the image of each edge of the square.

S_1 : (x, y) , $x = 2$, $-2 \leqslant y \leqslant 2$

Then

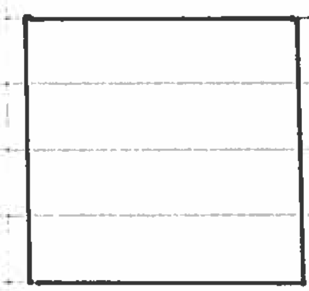
$$T\begin{bmatrix} x \\ y \end{bmatrix} = T\begin{bmatrix} 2 \\ y \end{bmatrix} = \begin{bmatrix} \quad \end{bmatrix}\begin{bmatrix} \, \end{bmatrix} = \begin{bmatrix} \quad \end{bmatrix} = \begin{bmatrix} X \\ Y \end{bmatrix}$$

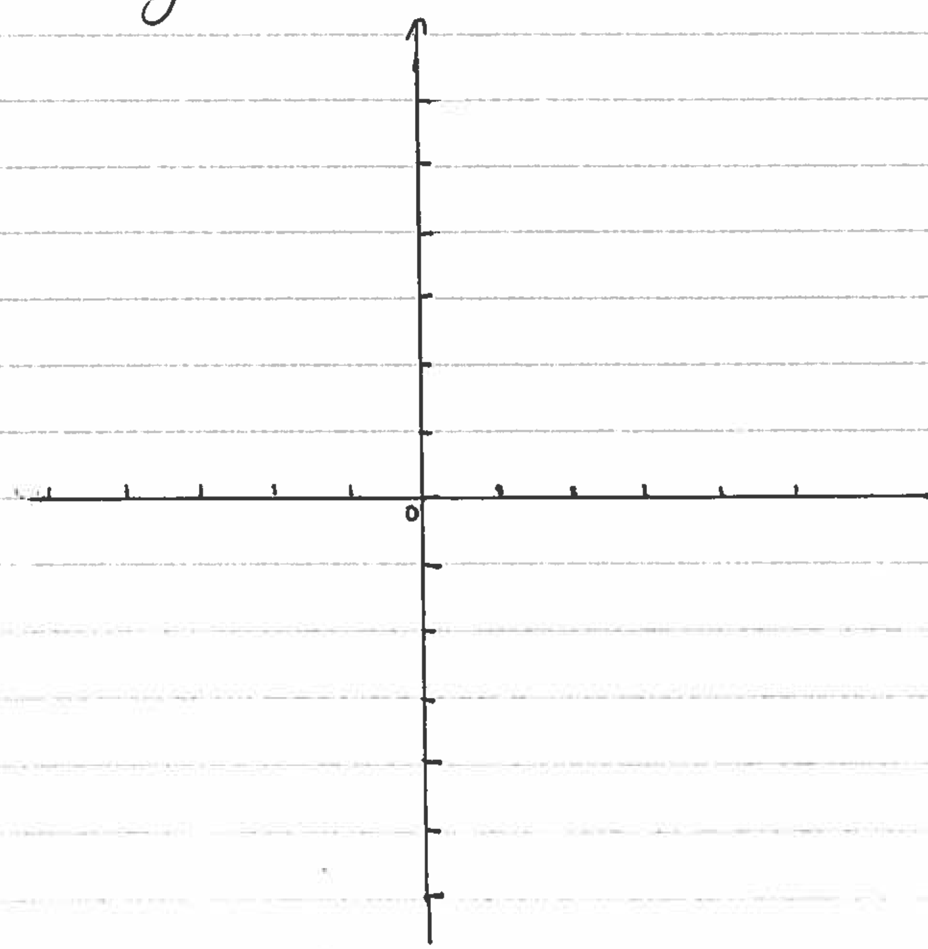
So $X =$, $y =$

$Y =$ $=$ $=$

Hence the image of the point $(x, y) = (2, y)$ lies on the line

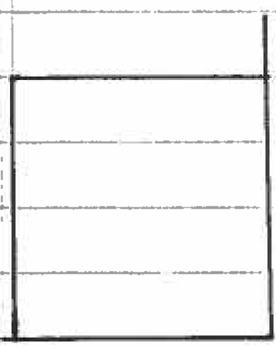
Also since __ $\leqslant y \leqslant$ __ , __ $\leqslant X \leqslant$ __

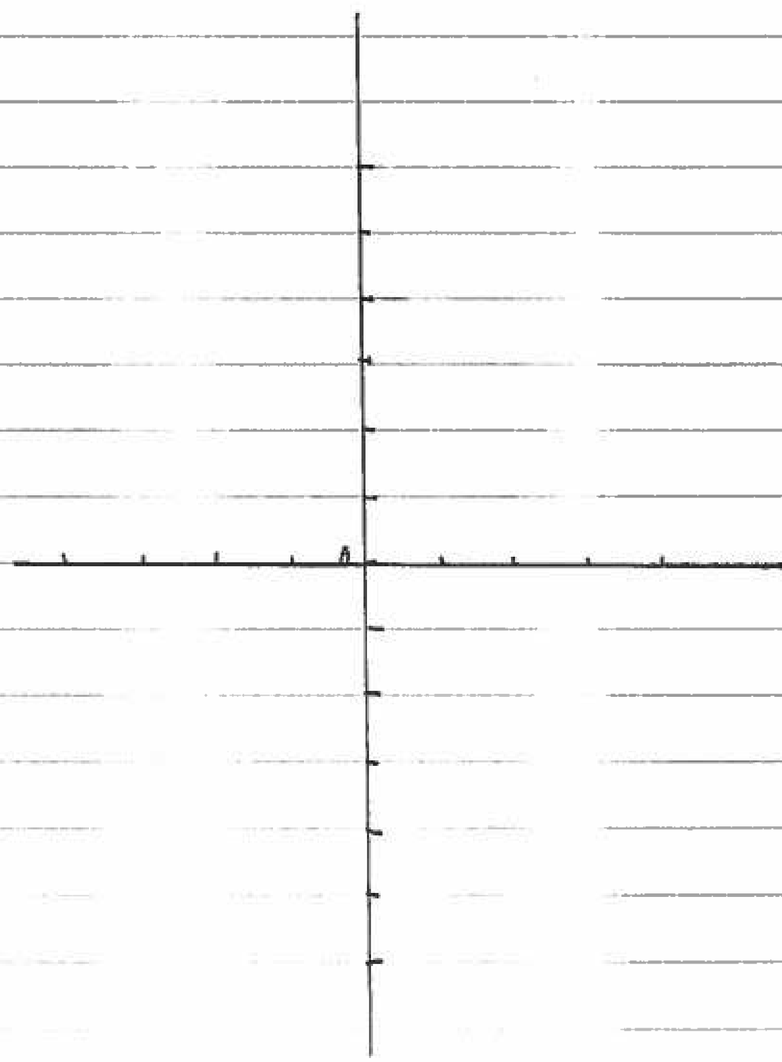




(p. 60)

Vertex (a, b)	Image $T(a, b)$
$(2, 2)$	
$(2, -2)$	
$(-2, 2)$	
$(-2, -2)$	





__Definition__ A transformation $T: \mathbb{R}^n \to \mathbb{R}^m$ is linear if

①

and ②

__NOTE:__ Every matrix transformation is linear.

__Proof.__ Let A be an $m \times n$ matrix and define $T: \mathbb{R}^n \to \mathbb{R}^m$ by $T(\vec{x}) = A\vec{x}$.
Let $\vec{u}, \vec{v} \in \mathbb{R}^n$ and $c \in \mathbb{R}$. Then

$T(\vec{u} + \vec{v}) =$

$=$

$T(c\vec{u}) =$

$=$

Thus $\square$

__Proposition__ Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be linear. Then

① $T(\vec{0}) =$

② $T(\alpha \vec{u} + \beta \vec{v}) =$

for

③ $T(\alpha_1 \vec{u_1} + \alpha_2 \vec{u_2} + \cdots + \alpha_k \vec{u_k}) =$

for

Proof ① $T(\vec{0}) =$

$=$

$=$

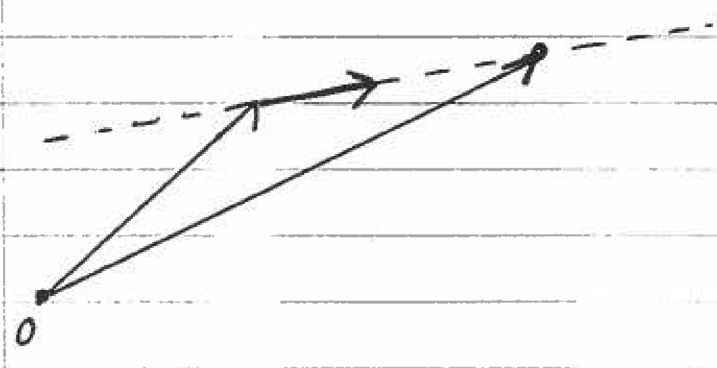
② Let $\vec{u}, \vec{v} \in \mathbb{R}^n$ & $\beta, \alpha \in \mathbb{R}$. Then

$T(\alpha\vec{u} + \beta\vec{v}) =$

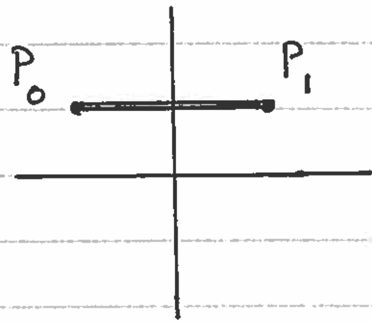
$=$

Problem (#25) Given $\vec{v} \neq \vec{0}$, $\vec{p} \in \mathbb{R}^n$, the line through $\vec{p}$ with direction vector $\vec{v}$ has the parametric equation $\vec{x} = \vec{p} + t\vec{v}$.

Show that a linear transformation $T: \mathbb{R}^n \to \mathbb{R}^m$ maps this line onto another line or a single point.



Example Let $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ by $T(\vec{x}) = A\vec{x}$ where $A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$. Find the image of the line segment from $(-2, 2)$ to $(2, 2)$ under T.



Section 1.9 The Matrix of a Linear Transformation

Theorem Suppose $T: \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation. Then there exists a ____ __ __ matrix A such that

$$T(\vec{x}) =$$

for . In fact

$$A = \left[\quad \Big| \quad \Big| \cdots \Big| \quad \right]$$

where

$$\vec{e}_1 = \begin{bmatrix} \\ \\ \\ \end{bmatrix}, \quad \vec{e}_2 = \begin{bmatrix} \\ \\ \\ \end{bmatrix}, \quad \dots, \quad \vec{e}_n = \begin{bmatrix} \\ \\ \\ \end{bmatrix}$$

are the ____ of the $n \times n$ __ __ __ matrix __.

Proof: Suppose $T: \mathbb{R}^n \to \mathbb{R}^m$ is linear.

Let

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

Then $\vec{x} =$

$$T(\vec{x}) =$$

$$=$$

$=$

$= A\vec{x},$

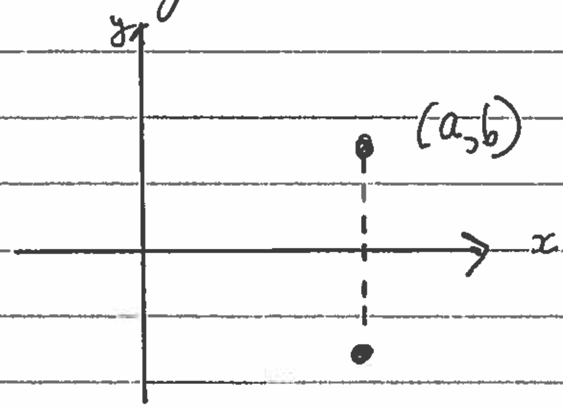
where $A = \left[\quad \middle| \quad \middle| \cdots \middle| \quad \right]$

is an __ $\times$ __ matrix since it has __ columns and each column is a vector in ___.

Uniqueness is left as an exercise. $\square$

<u>NOTE</u> The matrix A is called the ____ __ matrix of the ________.

<u>Example</u> Show that reflection in the x-axis is a linear transformation & find the standard matrix of this transformation.



Let $\vec{x} = \begin{bmatrix} \\ \end{bmatrix}$. Then $T(\vec{x}) = \begin{bmatrix} \\ \end{bmatrix}$

$$T(\vec{x}) = \begin{bmatrix} \\ \\ \end{bmatrix} + \begin{bmatrix} \\ \\ \end{bmatrix}$$

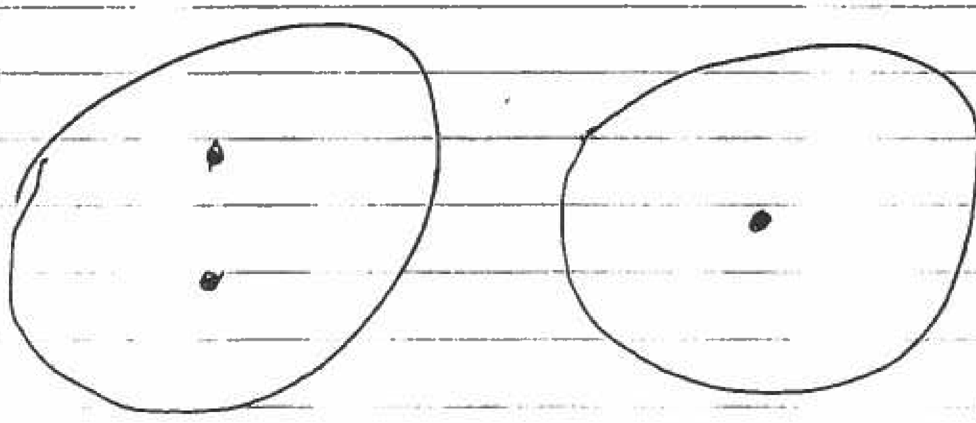
$$= \begin{bmatrix} & \\ & \\ \end{bmatrix}\begin{bmatrix} \\ \\ \end{bmatrix}$$

So
$T(\vec{x}) = \_\_\_\_$ where $A =$

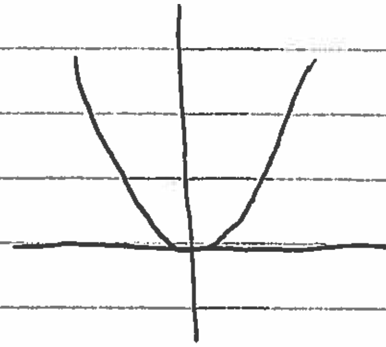
Hence $T: \_\_\_ \to \_\_\_$ is a linear transformation and the standard matrix of T is

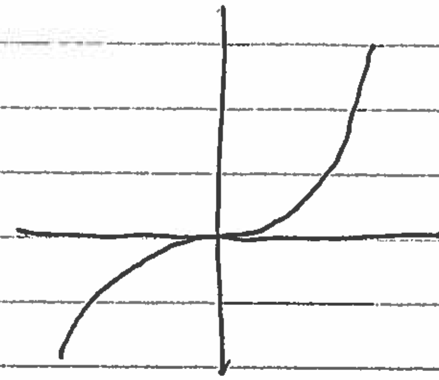
$$A = \begin{bmatrix} \\ \\ \end{bmatrix}.$$

Definition: A function $f: A \to B$ is one-to-one if ________ implies ____.



(p. 67)





__Theorem__ Let $T:\mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a linear transformation. Then T is one-to-one iff the equation

$$(*) \qquad\qquad = $$

has ________________.

__Proof__ Let $T:\mathbb{R}^n \longrightarrow \mathbb{R}^m$ be linear.

($\Longrightarrow$) Suppose T is one-to-one.

Suppose $T(\vec{x}) = \vec{0}$ where ______.

We know

$$T(\quad) = $$

since T is ________.

Therefore

$$T(\quad) = T(\quad).$$

This implies $\vec{x} =$ ___ since T is ________.

Hence $(*)$ has ________________.

($\Leftarrow$) Suppose (*) has ________.
Suppose $\vec{x}_1, \vec{x}_2 \in \mathbb{R}^n$ and
$$T(\vec{x}_1) = T(\vec{x}_2).$$
Then
$$T(\vec{x}_1 - \vec{x}_2) =$$
$$=$$
$$=$$

So $\vec{x} =$ ________ is a solution (*).
Hence $\vec{x}_1 =$ since the ___ solution
solution of (*) is $\vec{x} =$ ___.
Therefore
T is ________ $\square$

Example Determine whether the following linear transformations are one-to-one.

(i) $T: \mathbb{R}^2 \to \mathbb{R}^2$ by $T(\vec{x}) = A\vec{x}$ where $A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$.

(ii) $U: \mathbb{R}^2 \to \mathbb{R}^2$ by $U(\vec{x}) = B\vec{x}$ where $B = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$.

(i) $T(\vec{x}) = A\vec{x} = \vec{c}$

$\Leftrightarrow \begin{bmatrix} & \\ & \end{bmatrix} \sim \begin{bmatrix} & \\ & \end{bmatrix} \sim \begin{bmatrix} & \\ & \end{bmatrix}$

and so T is ________.

(ii) $U(\vec{x}) = B\vec{x} = \vec{0}$ (*)

$$\Leftrightarrow \quad \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix} \sim \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}$$

which has ________ since ________

So (*) has ________ and U is ________.

In fact

$$U\left(\begin{bmatrix} \quad \\ \quad \end{bmatrix}\right) = U\left(\begin{bmatrix} \quad \\ \quad \end{bmatrix}\right) = \vec{0}$$

Theorem Let A be an $m \times n$ matrix and suppose $T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ by $T(\vec{x}) = A\vec{x}$. Then T is one-to-one iff the ________ of A are ________.

PROOF

T is one-to-one $\Leftrightarrow$ $T(\vec{x}) =$ ___ has ________ ________

$\Leftrightarrow$ $A\vec{x} =$ has ________

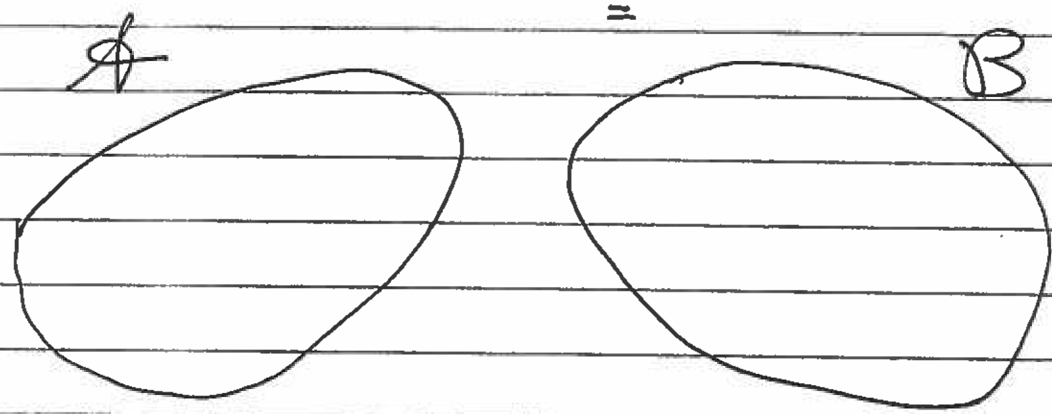
$\Leftrightarrow$ The ________ of A are ________

$\square$

NOTE: The columns of A are .... iff A has a ________.

Definition A function $f: A \to B$ is **onto** if for ______ there is a ____ such that

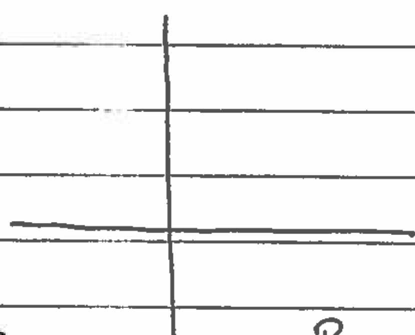
____ = ____

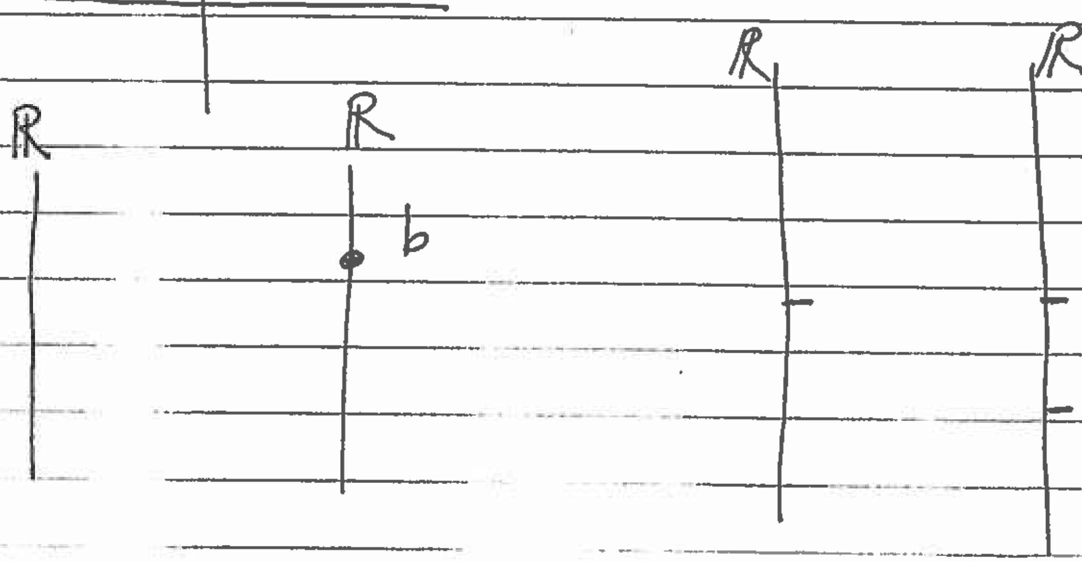


Examples

$f: \mathbb{R} \to \mathbb{R}$ by $f(x) = 2x$ is ....

$g: \mathbb{R} \to \mathbb{R}$ by $g(x) = x^2$ is ....





Theorem Let A be an $m \times n$ matrix and define $T: \mathbb{R}^n \to \mathbb{R}^m$ by $T(\vec{x}) = A\vec{x}$.
Then T is onto iff ________.

NOTE: The ______ of A ______ iff
A has a ______ in ________.

PROOF: T is onto $\iff$ For all $\vec{b} \in$ __.
$T(\ \ ) =$
for ______.

$\iff$ $A\vec{x} =$ is ________.

$\iff$ Columns of A ______. $\square$

Example Let $T: \mathbb{R}^2 \to \mathbb{R}^3$ by

$$T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3x_1 + x_2 \\ 5x_1 + 7x_2 \\ x_1 + 3x_2 \end{pmatrix}$$

(i) Determine whether T is one-to-one.
(ii) Determine whether T is onto.