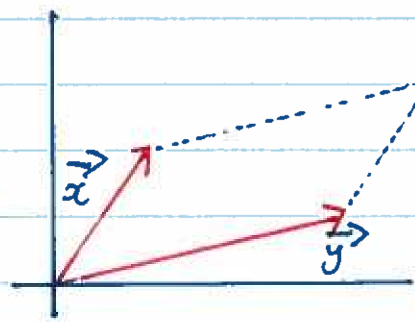
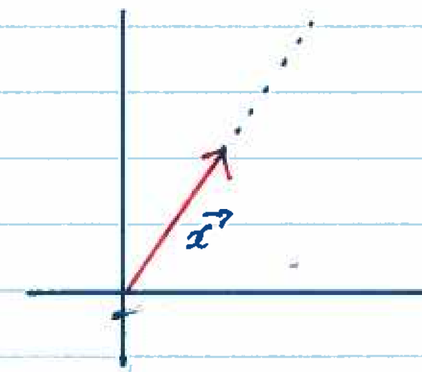
Chapter 1 Vector Spaces

1.1 Introduction

Addition & Scalar Multiplication of Vectors in the Plane





Properties

1. For all vectors $\vec{x}, \vec{y}$, $\vec{x} + \vec{y} =$
2. for all vectors $\vec{x}, \vec{y}, \vec{z}$, $(\vec{x} + \vec{y}) + \vec{z} =$
3. There exists a vector $\vec{0}$ such that

4. For each vector $\vec{x}$ there is a vector $\vec{y}$ such that
5. For each vector $\vec{x}$, $1\vec{x} =$
6. For any real numbers a, b and any vector $\vec{x}$
   $(ab)\vec{x} =$
7. For any real number a and any vectors $\vec{x}, \vec{y}$,
   $a(\vec{x} + \vec{y}) =$
8. For any real numbers a, b and any vector $\vec{x}$
   $(a+b)\vec{x} =$

1.2 Vector Spaces

Definition A vector space V over a field F is a set V (of mathematical objects) with two operations (__________ and ____________) defined so that for each pair $\vec{u}, \vec{v}$ in V there is a ________ element ______ in ____ and for each __________ and each _____ there is a unique element ____________. such that the following conditions hold:

(VS1) For all $\vec{u}, \vec{v}$ in V, $\vec{u} + \vec{v} =$ ______.

(VS2) For all $\vec{u}, \vec{v}, \vec{w}$ in V, $\vec{u} + (\vec{v} + \vec{w}) =$ ________.

(VS3) There exists an element ___ in V such that $\vec{u} +$ ___ $= \vec{u}$ for _______.

(VS4) For each _______ there exists ______ such that _______ $= \vec{0}$.

(VS5) For each $\vec{u} \in V$, $1\vec{u} =$ ____.

(VS6) For all $a, b \in$ ___ and $\vec{u} \in V$, $(ab)\vec{u} =$ _____.

(VS7) For all $a \in$ ___ and $\vec{u}, \vec{v} \in V$, $a(\vec{u} + \vec{v}) =$ ________.

(VS8) For all $a, b \in$ ___ and ____.
$(a+b)\vec{u} =$ ______.

NOTE
(1) The definition of a field is given in ____.
(2) In our course F is usually __ or __.
(3) The elements of F are called ____.
The elements of V are called _____.
(4) [PROOF LATER] The vector $\vec{v}$ in (VS4) is ____
and is denoted by ____.
(5) [PROOF LATER] The vector $\vec{0}$ is ____.

Examples of Vector Spaces
(1) Let F be a field and n be a positive integer.
Define

$F^n :=$

_ _ _ _ _ _ _ _ _ _ ..

Let
$\vec{x} =$, $\vec{y} =$ $\in F^n$ & $a \in F$.
Define

$\vec{x} + \vec{y} :=$, $a\vec{x} =$

Then with these operations F^n is a ____
over ___.

<u>Example</u> $\mathbb{R}$ is a ____.

$\mathbb{R}^4 =$

is a ___ over ___.

(2) Let F be a field.
Let $P(F) =$ ....

Then $P(F)$ is a vector space over __ with addition and scalar multiplication defined in the usual way.

<u>Example</u> Let

$p_1(x) =$

$p_2(x) =$ $\qquad\qquad\qquad\qquad\qquad \in$

Then $p_1(x) + p_2(x) =$

$3\, p_1(x) =$

<u>Theorem (Cancellation Law for Vector Addition).</u>
Let V be a vector space over a field F.
If $\vec{u}, \vec{v}, \vec{w} \in V$ and

$$\vec{u} + \vec{w} = \_\_\_\_\_$$

Then $\vec{u} =$ ___.

Proof: Suppose $\vec{u}, \vec{v}, \vec{w} \in V$ and

(*) $\vec{u} + \vec{w} = \vec{v} + \vec{w}$. Then

$\vec{u} = \vec{u} + \vec{0}$ (by ______),

$=$ ______________ (by ______ for ______________)

$=$ ______________ (by ______)

$=$ ______________ (by (*))

$=$ ______________ (by ______)

$=$ ______________ (   )

$=$ __________ (by ______).

Thus

$\vec{u} =$ ______. $\square$

Corollary 1 The vector $\vec{0}$ in (VS3) is ________.

Proof We know that

(A) $\vec{u} + \vec{0} =$ ______ for ____________.

Suppose $\vec{z} \in V$ and

(B) ______ $=$ ______ for ____________.

We will show ______ $=$ ______.

By (A), $\vec{0} + \vec{0} = $ ___.

By (B) $\vec{0} + \vec{z} = $ ___.

Hence

$$\vec{0} + \vec{0} = \_\_\_,$$

and

$$\vec{0} + \vec{z} = \_\_\_ \quad (\text{by } \_\_\_)$$

Hence $\vec{0} = $ ___. by Cancellation Law.

Thus the vector ___________. $\square$

<u>Corollary 2</u> The vector $\vec{v}$ in (VS 4) is ____.

<u>Proof</u>. Suppose V is a vector space over F, $\vec{u} \in V$ and

$$\vec{u} + \vec{v_1} =$$

$$\vec{u} + \vec{v_2} =$$

for ________.

We will show ___ = ___.

Then

$$\vec{v_1} + \vec{u} = \_\_\_ = \_\_\_,$$

$$\vec{v_2} + \vec{u} = \_\_\_ = \_\_\_.$$

Thus

$$\vec{v_1} + \vec{u} = \_\_\_ \quad \text{and}$$

$$\vec{v_1} =$$

by the ________. Therefore the vector $\vec{v}$ in (VS 4) is ______. $\square$

More Examples

(3) Let m, n be positive integers and F be a field.
An $m \times n$ matrix with entries from F has the form

$$\begin{array}{cc} A = \begin{pmatrix} & & \\ & & \\ & & \\ & & \end{pmatrix} & \leftarrow i\text{th row} \\ \uparrow & \\ j\text{th column} & \end{array}$$

where each ___ $\in$ ___.
The set of such matrices is denoted by ___.
Let $A, B \in$ ___ and $c \in$ ___.
Let $(A)_{ij}$ denote ____.
We define $A+B$, $cA \in$ ___ by

$(A+B)_{ij} =$

$(cA)_{ij} =$

for $1 \leq i \leq$ __ and $1 \leq j \leq$ ___.
Then ___ is a vector space over ___.

<u>Example</u>

$$M_{3\times 2}(\mathbb{R}) = \left\{ \qquad\qquad\qquad\qquad \right\}.$$

Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}$, $B = \begin{pmatrix} 2 & -1 \\ -5 & -6 \\ 0 & 7 \end{pmatrix}$.

Then

$A + B =$ $\qquad\qquad\qquad$, $2A =$ $\qquad\qquad$.

(4) Let S be a nonempty & F be a field.
Let
$\mathcal{F}(S, F)$ be the set ____________.

[For example if $S = \{x \in \mathbb{R} : x > 0\}$, $F = \mathbb{R}$
$f(x) =$ $\qquad\qquad$, $g(x) =$ $\qquad\qquad$ (for $x \in$ ___).
Then
$f, g \in \mathcal{F}(S, F) =$ ________].

For $f, g \in \mathcal{F}(S, F)$ define $f + g$ by

$(f + g)(s) :=$

and cf by $\qquad$ (where $c \in$ $\qquad$) by

$(cf)(s) :=$

Under these operations $\mathcal{F}(S, F)$ is a ____________.

NOTE:

(1) $P(F) \subset$ ________.

(2) The zero vector in $\mathcal{F}(S, F)$ is the ____ function $z(x) =$ for ____.

Theorem Let V be a vector space over a field F.

(1) If $\vec{u} \in V$ then $0\vec{u} =$ ____.

(2) If $\vec{u} \in V$ and $a \in F$ then

$$(-a)\vec{u} = -(\_\_\_\_) = a(\_\_\_\_).$$

(3) If $a \in F$ then $a\vec{0} =$ ____.

PROOF Suppose V is a vector space over a field F.

(1) Suppose $\vec{u} \in V$. Since F is a field we know

__________.

$0\vec{u} + 0\vec{u} =$ ________ (by ____)

$=$ ____ (since)

$=$ ________ (by)

Thus

$0\vec{u} + 0\vec{u} =$ ________,

and

$0\vec{u} =$ ____

by the __________. $\square$

(2) Let $\vec{u} \in V$, $a \in F$.

$a\vec{u} + (-a)\vec{u} =$

$=$

$=$

Hence $(-a)\vec{u} =$

by

Letting $a = 1$ we have

(*) $(-1)\vec{u} =$ (by - - -)

for any - - - .

So

$-(a\vec{u}) =$

$=$

$=$

since in a field $(-1)a =$

(3) Let $a \in F$. Then

$$\vec{0} + \vec{0} = \qquad\qquad (\text{by} \qquad)$$

Thus

$$a(\vec{0} + \vec{0}) =$$

But

$$a(\vec{0} + \vec{0}) = \qquad\qquad (\text{by} \qquad)$$

Hence

$$a\vec{0} + a\vec{0} =$$

$$= \qquad\qquad (\text{by} \qquad)$$

Thus

$$a\vec{0} + a\vec{0} =$$

and

$$a\vec{0} =$$

by the _ _ _ _ _ _ _ _ . □

Example

Let $S = \{ (a_1, a_2) : a_1, a_2 \in \mathbb{R} \}$.

For $(a_1, a_2), (b_1, b_2) \in S$ & $c \in \mathbb{R}$ define

$$(a_1, a_2) + (b_1, b_2) := (a_1 + b_1, a_2 - b_2),$$

$$c(a_1, a_2) := (ca_1, ca_2).$$

Show that S is not a vector space over $\mathbb{R}$.

1.3 Subspaces

Definition Let V be a vector space over a field F.
A ______ W of V is a subspace of V
if ______________________________
____

Theorem 1.3 Let V be a vector space over a field F and suppose W ___ V.
W is a subspace of V if and only if the following 3 conditions hold:

(1)

(2)

(3)

PROOF:

($\Rightarrow$) Suppose W is a subspace of V.
Then $W \subset V$ and W is a vector space.
So W must contain a zero vector $\vec{0} \in W \subset V$.

($\Leftarrow$) Suppose V is a vector space over a field F, $W \subset V$
and (1), (2), (3) hold.
Then addition & scalar multiplication
are _ _ _ _ _ _ _ _ _ _ by _ _ _, _ _ _.
Axioms (VS __), (VS __), (VS __), (VS __),
(VS 7), (VS 8) hold in W
since _ _ _ _ _ _ _ _ and they hold in V
since _ _ _ _ _ _ _ _.
Axiom (VS3) holds in W by _ _ _ _.
Axiom (VS4) holds in W since _ _ _ _ _ _ _.
_ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _
all axioms hold in W so that
_ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _.

__Example__
Show that $S = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 : x_1 + x_2 + x_3 = 0 \right\}$
is a subspace of $\mathbb{R}^3$.

Clearly
also $\vec{0} =$

Suppose $\vec{x} = \begin{bmatrix} \\ \\ \end{bmatrix}$, $\vec{y} = \begin{bmatrix} \\ \\ \end{bmatrix} \in$ _ _ _.

The Transpose of a Matrix

The transpose of a matrix A is obtained by ________________________ and is denoted by ____.

Example Let $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$

Then $A^t =$

NOTE: If A is $m \times n$ then A^t is ____ and

$$(A^t)_{ij} =$$

Ex Let $A, B \in M_{m \times n}(F)$, $a, b \in F$.
Prove that
$$(aA + bB)^t = aA^t + bB^t.$$

PROOF $\left((aA + bB)^t\right)_{ij} =$

Definition A matrix A is symmetric if _ _ _ _.

NOTE Symmetric matrices are square since $A^t = A$ implies _ _ _ _.

Example $A = \begin{pmatrix} 1 & 2 & 3 \\ & 4 & 5 \\ & & 6 \end{pmatrix}$

is symmetric since _ _ _ _.

Ex Let $W = \{A \in M_{n\times n}(F) : A = A^t\}$.
Prove that W is a subspace of ____.

PROOF:

Clearly

The zero

Suppose $A, B \in$ ___ and $c \in$ ___.

Ex. Let $W = \{p(x) \in P(F) : p(x) = x^2 g(x)$ for some $g(x) \in P(F)\}$.
Show that W is a subspace of ____.

Clearly

The zero

Let $p_1(x)$, $p_2(x) \in$ ___ and $c \in$ __.

<u>Example</u>

Let $W = \left\{ \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \in M_{3\times3}(\mathbb{R}) : a_{11}+a_{22}+a_{33}=1 \right\}$

Determine whether W is a subspace of $M_{3\times3}(\mathbb{R})$.

Example

Let $W = \left\{ A \in M_{3\times3}(\mathbb{R}) : A_{11} + A_{22} + A_{33} = 0 \right\}$.

Determine whether W is a subspace of $M_{3\times3}(\mathbb{R})$ and prove it.

Theorem Let V_1, V_2 be subspaces of vector space V over F. Then V_1 ___ V_2 is a subspace of V.

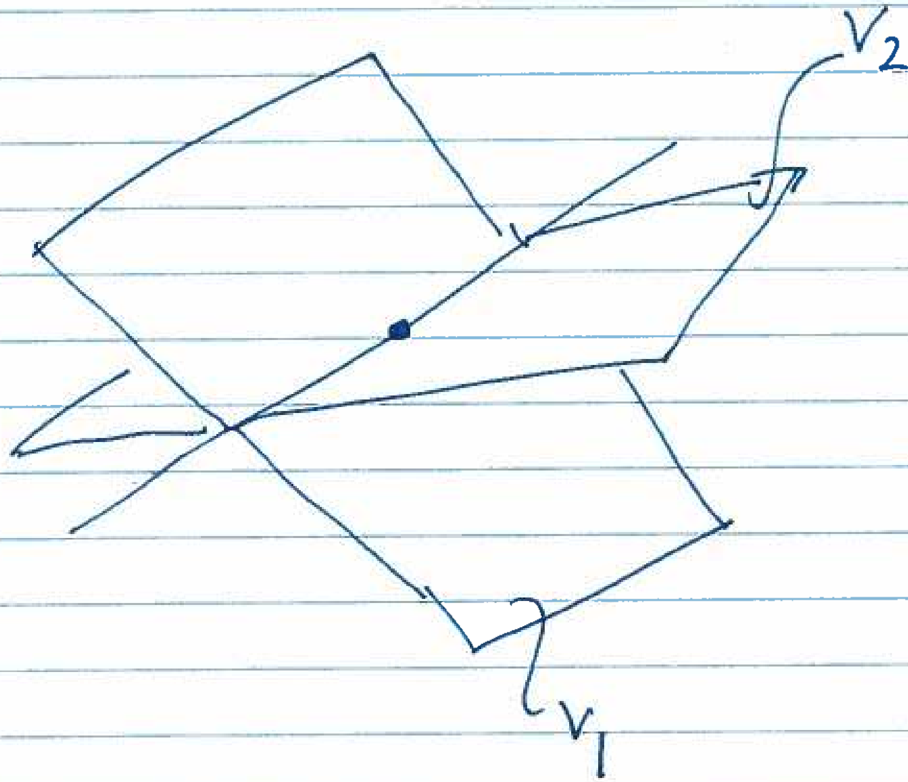
Proof:

Suppose V_1, V_2 are subspaces of vector space V over F.

Then V_1 , V_2 and so V_1 V_2 .

Let $\vec{0}$ be the zero vector of V.

. Suppose $\vec{u}, \vec{v} \in V_1 \cap V_2$ and $c \in F$.



An example in $\mathbb{R}^3$

Ex Is $V_1 \cup V_2$ a subspace of V if V_1, V_2 are subspaces of V?

Theorem Suppose I is a set & V_α is a subspace of a vector space V over F for each $\alpha \in I$. Then

$$W =$$

is a subspace of ---.

NOTE $\vec{w} \in W$ iff

Definition

Let S_1, S_2 be non-empty subsets of a vector space V.
Then

$$S_1 + S_2 := \{ \qquad\qquad\qquad\qquad \}.$$

Example Let $S_1 = \left\{ \begin{bmatrix} a \\ a \\ -2a \end{bmatrix} : a \in \mathbb{R} \right\}$,

$S_2 = \left\{ \begin{bmatrix} b \\ 0 \\ b \end{bmatrix} : b \in \mathbb{R} \right\}$. Determine whether $\vec{u} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \in S_1 + S_2$?

Definition Let W_1, W_2 be subspaces of a vector space V over F.

V is the direct sum of W_1 and W_2 if

(1)

(2)

Note. In this case we write ___ __ __ .

Example

Let $W_1 = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 : x+y+z=0 \right\}$,

$W_2 = \left\{ \begin{bmatrix} a \\ a \\ a \end{bmatrix} : a \in \mathbb{R} \right\}$.

Show that $\mathbb{R}^3 = W_1 \oplus W_2$.

Theorem (Ex 23, p.22)

Let W_1, W_2 be subspaces of a vector space V over F.
Then

(i) $W_1 + W_2$ is a _ _ _ _ _ _ that _ _ _ _ _ _ _ _ .

(ii) If W is a subspace of V that contains both W_1 and W_2 then _ _ _ _ _ _ _ _ _ .

Theorem (Ex 30, p.23)

Let W_1, W_2 be subspaces of a vector space V over F.
Then

$$V = W_1 \oplus W_2$$

if and only if every vector $\vec{v} \in V$ can be written
_ _ _ _ _ _ _ as

$$\vec{v} = \_\_ + \_\_$$

where _ _ _ _ _ _ _ _ _ _ _ _ _ _ .

(p.25)

1.4 Linear Equations and Systems of Linear Equations

Definition Let S be a non-empty subset of a vector space V (over F). We say $\vec{v} \in V$ is a *linear combination of vectors in S* if

________________ and ________________ such that

________ = ________________.

In this case, we say $\vec{v}$ is a ________________________ and ________________ are the ________________.

Example Determine whether the vector $\vec{v} = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} \in \mathbb{R}^3$ is a linear combination of

$$\vec{v}_1 = \begin{bmatrix} 4 \\ 5 \\ 1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 6 \\ 7 \\ 2 \end{bmatrix}.$$

Example Determine whether the vector $\vec{v} = \begin{bmatrix} 7 \\ 9 \\ 1 \end{bmatrix} \in \mathbb{R}^3$ is a linear combination of

$$\vec{v}_1 = \begin{bmatrix} 4 \\ 5 \\ 1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 6 \\ 7 \\ 2 \end{bmatrix}.$$

Example Determine whether the polynomial

$$p(x) = 5x^3 + x^2 + 3x - 5$$

is a linear combination of

$$p_1(x) = x^3 + 2x^2 + 3x - 4,$$
$$p_2(x) = x^3 - x^2 - x + 1.$$

Definition:

a linear system of m equations in the n unknowns $x_1, x_2, \ldots, x_n$ (over a field F) has the form

$$(*)\quad \begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \quad\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

where each $a_{ij}, b_j \in$ ___ and each variable $x_j \in$ ___.

(*) corresponds the augmented matrix

$$\left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right)$$

Operations that preserve the solution set for a linear system over F are

(1) Interchange two _____ (____)

(2) Multiply an equation by a ________ (____)

(3) Add a multiple of one ____________ (____)

THESE CORRESPOND TO OPERATIONS ON THE CORRESPONDING AUGMENTED MATRIX:

(1)

(2)

(3)

Linear system is in <u>reduced Echelon form</u> if

(1) The first nonzero coefficient in each equation is ___.

(2) If an unknown is the first unknown with a ____ coefficient in ____ equation, then that unknown occurs with a _______. in each of ____ ___ ____. The first such unknown in an equation is called ______ variable.

(3) The leading variable of each equation has a _____ subscript than leading variables of __ __ ____ equations.

The process of obtaining Reduced Echelon Form is called ________ - _________ <u>Elimination.</u>

<u>NOTE</u>: If we obtain an equation of the form $0 = c$ where $c \neq 0$, the system is called ________. and has __ solutions.

<u>Definition:</u> Let V be a vector space over F and suppose $\phi \neq S \subset V$. The <u>span of S</u> denoted by _______ is defined by

$$\text{Span}(S) = \{\vec{v} : \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\}$$

By convention we define $\text{Span}(\phi) = $ ______.

Example Let $\vec{v_1} = \begin{bmatrix} 4 \\ 5 \\ 1 \end{bmatrix}$, $\vec{v_2} = \begin{bmatrix} 6 \\ 7 \\ 2 \end{bmatrix} \in \mathbb{R}^3$.

(i) Is $\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} \in \mathrm{Span}(\{\vec{v_1}, \vec{v_2}\})$?

(ii) Is $\begin{bmatrix} 7 \\ 9 \\ 1 \end{bmatrix} \in \mathrm{Span}(\{\vec{v_1}, \vec{v_2}\})$?

Example In the vector space $P(\mathbb{R})$ let

$p(x) = 5x^3 + x^2 + 3x - 5$,

$p_1(x) = x^3 + 2x^2 + 3x - 4$,

$p_2(x) = x^3 - x^2 - x + 1$,

$S = \{p_1(x), p_2(x)\}$.

Determine whether $p(x)$ is in the Span of S.

Theorem 1.5

Let V be a vector space over F and suppose $S \subset V$.
Then

(1) $Span(S)$ is a ________ of V.

(2) If W is a subspace of V and $S \subset W$
then $Span(S)$ ______.

PROOF:

(1) Case 1 $S = \phi$.

Case 2. $S \neq \phi$.

(2) Now suppose $W \subset V$ and W is a subspace of V,

<u>Definition</u> Let $S \subset V$ where V is a vector space.
We say S spans V (or _ _ _ _ _ _) V if
_ _ _ _ _ _ _ _.

<u>NOTE</u>
(1) $S \subset V$ so $\operatorname{Span}(S)$ _ _ _ _ _.
Hence $\operatorname{Span}(S) = V$ iff _ _ _ _ _ _;
ie iff each vector of _ _ _ _ _ _ _ _ _ _ _ _
_ _ _ _ _ _ _ _ _ _ _ _ _ _.
(2) In this case (see defn) we say vectors in S
_ _ _ _ _ _ _ _.

Example Show that the vectors

$$\vec{v_1} = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}, \quad \vec{v_2} = \begin{bmatrix} 1 \\ 0 \\ -4 \end{bmatrix}, \quad \vec{v_3} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \quad \text{in } \mathbb{R}^3$$

generated $\mathbb{R}^3$.

Example Show that $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $C = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ generate the vector space

$$V = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_{2\times 2}(\mathbb{R}) : a+d=0 \right\}.$$

<u>Definition</u> Two linear systems are <u>equivalent</u> if ________________.

We notation $\Longleftrightarrow$ to mean ________;

ie have ________________.

<u>Example</u> <u>Solve the linear system</u>

$$\begin{cases} 3x_2 - 6x_3 + 6x_4 + 4x_5 = -5 \\ 3x_1 - 7x_2 + 8x_3 - 5x_4 + 8x_5 = 9 \\ 3x_1 - 9x_2 + 12x_3 - 9x_4 + 6x_5 = 15 \end{cases} \qquad \left(\begin{array}{ccccc|c} 0 & 3 & 6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{array}\right)$$

$$\overset{\frac{1}{3}R_3}{\Longleftrightarrow}$$

$$\begin{cases} 3x_2 - 6x_3 + 6x_4 + 4x_5 = -5 \\ 3x_1 - 7x_2 + 8x_3 - 5x_4 + 8x_5 = 9 \\ x_1 - 3x_2 + 4x_3 - 3x_4 + 2x_5 = 5 \end{cases} \qquad \left(\begin{array}{c|c} \quad & \quad \end{array}\right)$$

$$\overset{\_\_\_\_}{\Longleftrightarrow}$$

$$\begin{cases} x_1 - 3x_2 + 4x_3 - 3x_4 + 2x_5 = 5 \\ 3x_1 - 7x_2 + 8x_3 - 5x_4 + 8x_5 = 9 \\ 3x_2 - 6x_3 + 6x_4 + 4x_5 = -5 \end{cases} \qquad \left(\begin{array}{ccccc|c} 1 & -3 & 4 & -3 & 2 & 5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{array}\right)$$

$$\overset{\_\_\_\_}{\Longleftrightarrow}$$

$$\begin{cases} x_1 - 3x_2 + 4x_3 - 3x_4 + 2x_5 = 5 \\ 2x_2 - 4x_3 + 4x_4 + 2x_5 = -6 \\ 3x_2 - 6x_3 + 6x_4 + 4x_5 = -5 \end{cases} \qquad \left(\begin{array}{ccccc|c} 1 & -3 & 4 & -3 & 2 & 5 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{array}\right)$$

$$\overset{\_\_\_\_}{\Longleftrightarrow}$$

$$\begin{cases} x_1 - 3x_2 + 4x_3 - 3x_4 + 2x_5 = 5 \\ x_2 - 2x_3 + 2x_4 + x_5 = -3 \\ 3x_2 - 6x_3 + 6x_4 + 4x_5 = -5 \end{cases} \qquad \left(\begin{array}{ccccc|c} 1 & -3 & 4 & -3 & 2 & 5 \\ 0 & 1 & -2 & 2 & 1 & -3 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{array}\right)$$

(P.37)

$$\Leftrightarrow \begin{cases} x_1 - 3x_2 + 4x_3 - 3x_4 + 2x_5 = 5 \\ \quad x_2 - 2x_3 + 2x_4 + x_5 = -3 \\ \quad\quad x_5 = 4 \end{cases} \quad \left(\begin{array}{ccccc|c} 1 & -3 & 4 & -3 & 2 & 5 \\ 0 & 1 & -2 & 2 & 1 & -3 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{array}\right)$$

$$\Leftrightarrow \begin{cases} x_1 - 3x_2 + 4x_3 - 3x_4 = -3 \\ \quad x_2 - 2x_3 + 2x_4 = -7 \\ \quad\quad x_5 = 4 \end{cases} \quad \left(\begin{array}{ccccc|c} 1 & -3 & 4 & -3 & 0 & -3 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{array}\right)$$

$$\Leftrightarrow \begin{cases} x_1 - 2x_3 + 3x_4 = -24 \\ \quad x_2 - 2x_3 + 2x_4 = -7 \\ \quad\quad x_5 = 4 \end{cases} \quad \left(\begin{array}{ccccc|c} 1 & 0 & -2 & 3 & 0 & -24 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{array}\right)$$

$$\Leftrightarrow \begin{cases} \\ \\ \end{cases}$$

where

The solution set

$S = \{$

1.5 Linear Dependence & Linear Independence (p.38)

Definition: Let $S \subset V$ where V is a vector space over F. S is linearly dependent if ________________ ____________ in ___ and _________ ______________ such that

(*) ____________________________ = _________

Otherwise we say S is ____________________.

NOTE:

(1) If S is linearly dependent then S is ________.
Thus $\phi = \{\ \}$ is ______________.

(2) $S = \{\vec{0}\}$ is ________________
since ________________________.

(3) If $\vec{u} \neq$ ___ then $S = \{\vec{u}\}$ is ____________.

PROOF: Suppose $\vec{u} \neq$ ___

Suppose

Example Let

$$S = \left\{ \vec{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ 3 \\ -2 \\ 1 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 3 \\ 4 \\ -2 \\ 3 \end{bmatrix}, \vec{v}_4 = \begin{bmatrix} 2 \\ 2 \\ -1 \\ 2 \end{bmatrix} \right\}$$

Determine whether the set S is linearly dependent or independent in $\mathbb{R}^4$.

Consider the vector equation

(*) $x_1 \vec{v_1} + x_2 \vec{v_2} + x_3 \vec{v_3} + x_4 \vec{v_4} = \_\_\_$

where $x_1, x_2, x_3, x_4 \in \_\_$

$$\Longleftrightarrow \quad x_1 \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 3 \\ -2 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ 4 \\ -2 \\ 3 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ 2 \\ -1 \\ 2 \end{bmatrix} =$$

$$\Longleftrightarrow \quad \begin{cases} x_1 + x_2 + 3x_3 + 2x_4 = 0 \\ -x_1 + 3x_2 + 4x_3 + 2x_4 = \_\_ \\ x_1 - 2x_2 - 2x_3 - x_4 = \_\_ \\ x_1 + x_2 + 3x_3 + 2x_4 = \_\_ \end{cases}$$

$$\Longleftrightarrow \quad \left[\begin{array}{cccc|c} 1 & & & & \\ -1 & & & & \\ 1 & & & & \\ 1 & & & & \end{array}\right]$$

$$\Longleftrightarrow \quad \left[\begin{array}{cccc|c} 1 & & & & \\ 0 & & & & \\ 0 & & & & \\ 0 & & & & \end{array}\right]$$

$$\Longleftrightarrow \quad \left[\begin{array}{cccc|c} 1 & 1 & & & \\ 0 & 1 & & & \\ 0 & -3 & & & \\ 0 & 0 & & & \end{array}\right]$$

$$\Leftrightarrow \left[\begin{array}{cccc|c} 1 & 1 & 3 & 2 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \Leftrightarrow \begin{array}{l} x_1 = \\ x_2 = \\ x_3 = \end{array}$$

This system is ______.

__ is a _____ variable.

So the system has ______ solutions and hence S is linearly ______.

One sol$\underline{n}$ is $x_4 =$, $x_3 =$, $x_2 =$, $x_1 =$

So $\qquad\qquad\qquad =$

and observe that

$$\vec{v}_4 = \_\_\_\_\_\_.$$

<u>Proposition</u> Let V be a vector space over F and suppose $S \subset V$. Then

S is linearly dependent if and only if there is a vector in S that is ________

______________... or __ $\in S$.

<u>PROOF</u>

($\Rightarrow$) Suppose S is linearly dependent.

Then

<u>Case 1</u> $n=1$.

<u>Case 2</u> $n>1$. Then

$$c_1 \vec{v}_1 =$$

$$\vec{v}_1 =$$

and $\vec{v}_1$ is

($\Leftarrow$)

<u>Case 1</u> Suppose $\vec{v}_1 \in S$ is a ______________.

<u>Case 2</u> Suppose $\vec{0} \in S$.

Theorem 1.6 Let V be a vector space over F and suppose $S_1 \subset S_2 \subset V$.

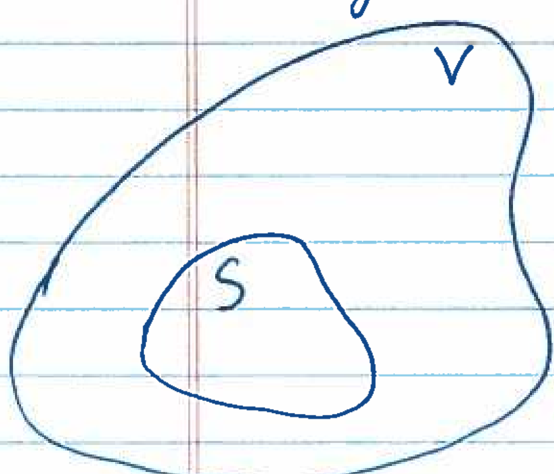
If S_1 is linearly _________ then S_2 is linearly ________.

PROOF: Suppose $S_1 \subset S_2 \subset V$, V is a vector space over F & S_1 is linearly dependent. So there are

Corollary Suppose V is a vector space over F & $S_1 \subset S_2 \subset V$.

If S_2 is linearly _________ then S_1 is linearly _________.

Theorem 1.7 Let S be a linearly independent subset of a vector space V.



Let $\vec{v} \in V \setminus S$

(ie _____________).

Then

_________________ is linearly dependent if and only if $\vec{v} \in$ _______.

PROOF. Assume S is a linearly independent subset of a vector space V and $\vec{v} \in V \setminus S$.

($\Rightarrow$) Suppose __________ is linearly dependent.
Then there are ______ vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n \in$ _____
and scalars $c_1, c_2, \ldots, c_n$ ________ such that
______________ = _____.

Since S is linearly independent at __________________. We may assume that (with loss of generality) that ______ and _____.

Case 1 $n=1$. Then

Case 2 $n>1$. Then
$c_1 \vec{v}_1 =$ and
$\vec{v} =$ $\in$ ________.
since ______________.

In both cases $\vec{v} \in$ ______.

($\Leftarrow$) Suppose $\vec{v} \in$ ______.

Example Let

$$S = \left\{ \vec{v_1} = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \vec{v_2} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \vec{v_3} = \begin{bmatrix} -1 \\ 2 \\ 2 \\ -1 \end{bmatrix}, \vec{v_4} = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 2 \end{bmatrix} \right\}$$

Determine whether S is linearly dependent or independent in $\mathbb{R}^4$.

PROVING LINEAR INDEPENDENCE/DEPENDENCE

Let $S=\{\vec{v}_1,\vec{v}_2,\dots,\vec{v}_n\} \subset V$ where V is a vector space over F.

(*) To prove S is linearly independent assume

____

where ____________.

Then prove we must have

________________________.

(*) To prove S is linearly dependent Consider

(*) ____________________.

and

EITHER

________________________.

OR

show (*) has ____________________.

VECTORS IN $\mathbb{R}^3$

Linearly independent

Linearly dependent

Linearly independent

Linearly dependent

1.6 Bases and Dimension

Definition Let $\mathcal{B} \subset V$ where V is a vector space over F. Then $\mathcal{B}$ is a *basis* for V if

___ ___ ___ ___ and ___ ___ ___ ___ ___.

Theorem 1.8

Suppose $\mathcal{B} = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\} \subset V$ where V is a vecter space over F. $\mathcal{B}$ is a basis for if and only if ___ vector $\vec{v} \in V$ can be written

___ ___ as a ___ ___ ___.

(*) $\vec{v} =$

where ___ ___ ___ ie the scalars ___ ___ ___ are ___.

PROOF:

($\Longrightarrow$) Suppose $\mathcal{B} = \{\vec{v}_1, \ldots, \vec{v}_n\}$ is a basis for V.

Then $\mathcal{B}$

Sine $\mathcal{B}$ spans V each vecter $\vec{v} \in V$ can be written as a ___ ___.

$\vec{v} =$

where

Suppose

$\vec{v} =$

is

Then $\vec{v} - \vec{v} =$ ___ and

$= \vec{0}$?

Thus

Hence

so that the ______ is ____.

($\Leftarrow$) Suppose each $\vec{v} \in V$ can be written ____
as a ________ (*).
Then clearly V _____.
Now suppose

$$c_1 \vec{v_1} + c_2 \vec{v_2} + \cdots + c_n \vec{v_n} = \vec{0}$$

where each _____...
But

$$= \vec{0}$$

By ____ we have

$$c_1 = \quad , c_2 = \quad , \ldots , c_n = \_\_\_,$$

and $\mathcal{B}$ is ______.
Hence $\mathcal{B}$ is _________. $\square$

Examples

(1) ___ is a basis for $\{\vec{0}\}$ since
_________ and _______.

(2)

$$\mathcal{E} = \left\{ \vec{e_1} = \begin{bmatrix} \\ \\ \\ \end{bmatrix}, \vec{e_2} = \begin{bmatrix} \\ \\ \\ \end{bmatrix}, \ldots, \vec{e_n} = \begin{bmatrix} \\ \\ \\ \end{bmatrix} \right\}$$

is a basis for F^n where F is a field.

NOTE
$$(\vec{e_k})_j = \left\{ \begin{array}{l} \\ \\ \end{array} \right.$$

Here $(\vec{e_k})_j$ is the ___ component of $\vec{e_k}$

<u>PROOF</u>: Let $\vec{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \in F^n$

Then $\vec{a} = \sum_{k=1}^{n}$ ___ $\vec{e_k}$

since $(\vec{a})_j = \sum_{k=1}^{n}$ ______

= ________ = a_j

for each $1 \leq j \leq n$. Therefore

$\mathcal{E}$ _____ F^n.

Now suppose $\sum_{k=1}^{n} x_k \vec{e_k} = \vec{0}$

where each x_k _____. Then

$\left(\sum_{k=1}^{n} x_k \vec{e_k} \right)_j =$

$\sum_{k=1}^{n}$ = ,

and x_j = ,

for each $1 \leq j \leq n$. Hence $\mathcal{E}$ is ______.

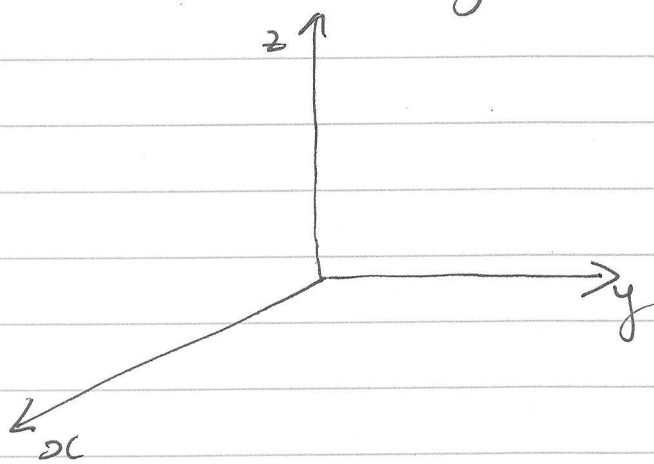
and $\mathcal{E}$ is a basis for F^n.

<u>Note</u>: $\mathcal{E}$ is called the ______ basis for F^n.

Example

$$\left\{ \vec{e}_1 = \begin{bmatrix} \\ \\ \\ \end{bmatrix}, \ \vec{e}_2 = \begin{bmatrix} \\ \\ \\ \end{bmatrix}, \ \vec{e}_3 = \begin{bmatrix} \\ \\ \\ \end{bmatrix} \right\}$$

is the standard basis for $\mathbb{R}^3$



$\vec{e}_1 =$

$\vec{e}_2 =$

$\vec{e}_3 =$

Example Determine whether the vectors

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \ \vec{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \ \vec{v}_3 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

form a basis for $\mathbb{R}^3$. Let $\vec{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \in \mathbb{R}^3$.

Consider the vector eq$\underline{n}$:

$$x_1 \vec{v}_1 + x_2 \vec{v}_2 + x_3 \vec{v}_3 = \vec{v}$$

$\Longleftrightarrow$

Now consider the vector eqn
$$x_1 \vec{v}_1 + x_2 \vec{v}_2 + x_3 \vec{v}_3 = \underline{\quad\quad}.$$

<u>Example</u>

Let $n \geq 1$ (integer), and

$P_n(F)$ = Set of polynomials in $P(F)$ with ______.

Then $P_n(F)$ is a subspace of ___ and

$\{ \_\_\_\_\_\_\_\_\_\_\_\_\_ \}$ is a basis for $P_n(F)$.

Also $\{ \quad\quad\quad \}$ is a basis for $P(F)$.

Example Find a basis for the vector space S of 2×2 symmetric matrices with real entries.
Here $F =$ ___ and
$S = \{ \qquad \qquad \}$.

Theorem 1.9 Let V be a vector space over a field F. Suppose $S \subset V$ and $Span(S) = V$ and suppose S is a finite set. Then __________.
is a basis for V, and V has a ____ basis.

PROOF: Suppose $S \subset V$, $Span(S) = V$ & S is a finite set.

Case 1 $S =$ ___ or ___.
Then $V =$ ____
and ___ is a basis for V.

<u>Case 2</u> $S \neq$ ___ and ___ <u>ie</u> S contains ________.

Then $\{\vec{v}\} \subset S$ and $\{\vec{v}\}$ is ________.

Consider all non-empty subsets $T \subset S$ which are ________.

Since S is finite choose a _____ $\mathcal{B} \subset S$ (<u>ie</u> ________) and $\mathcal{B}$ is ____________.

If $\mathcal{B} = S$ then

Now suppose $\mathcal{B} \subsetneq S$. We claim $S \subset$ ___

Suppose by way of contradiction that this claim ~~does not~~ does not hold; <u>ie</u> there is a ____ such that ___ __.

Then

Let $\mathcal{B}' =$ _____.

Then $\mathcal{B}'$ is _________ by Theorem 1.7 since _____.

But $|\mathcal{B}'|$

This contradicts ________.

Hence $S \subset$ ___, and

$V = \mathrm{Span}(S) \subset \qquad \subset V$

and $V =$

so that $\mathcal{B}$ is a ____ for V.

$\square$

<u>Example</u> Let

$$W = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \in \mathbb{R}^4 : x_1 + x_2 + x_3 + x_4 = 0 \right\}.$$

Let $S = \left\{ \vec{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 2 \\ -2 \\ 0 \\ 0 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 2 \\ -1 \\ -1 \\ 0 \end{bmatrix}, \right.$

$\left. \vec{v}_4 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \vec{v}_5 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} \right\}.$

It can be shown that

$$\mathrm{Span}(S) = W.$$

Find a subset $\mathcal{B}$ of S which is a basis for W.

Consider the vector eqn

$$x_1 \vec{v}_1 + x_2 \vec{v}_2 + x_3 \vec{v}_3 + x_4 \vec{v}_4 + x_5 \vec{v}_5 = \vec{0}$$

$\Leftrightarrow$

(p.58)

<u>Theorem 1.10</u> (<u>The Replacement Theorem</u>)

Let V be a vector space over F.

Suppose $\mathcal{G}$, $\mathcal{L}$ are finite subsets of V such that

$$V = \mathrm{Span}(\mathcal{G}), \quad |\mathcal{G}| = n,$$

$\mathcal{L}$ is linearly independent, and $|\mathcal{L}| = m$.

Then

m _____, and there is a set $\mathcal{H} \subset \mathcal{G}$

$|\mathcal{H}| =$ _____ and $\mathrm{Span}($_____$) = V$.

PROOF We proceed by induction on m. Let

If $m=0$ then then $\mathcal{L}=$ and $\wedge$ $\mathcal{H}=$

$m=0 \leqslant$ and $\text{Span}(\quad)=$

Suppose the statement is true for a fixed integer m.

Let $\mathcal{L}=\{\vec{v}_1, \ldots, \vec{v}_m, \vec{v}_{m+1}\}$ be ,

$V=$ and $|\mathcal{G}|=$.

Then $\mathcal{L}'=\{\vec{v}_1, \ldots\ldots, \}$ is

and by the _________. There is a subset

$\mathcal{X}'=\{ \quad \}$ of $\mathcal{G}$ such that

$\text{Span}(\quad)=\text{Span}(\{ \quad \})=$ ___.

Hence

$\vec{v}_{m+1}=$

NOTE: $n-m>$ ___ otherwise $\vec{v}_{m+1}$ would be a linear combination of which would contradict ___ being linearly ______.

Therefore $n-m \geqslant$ __ and $n \geqslant$ ___.

Similarly at least one of the ___________ is _____. otherwise

$\vec{v}_{m+1}=$

and ___ is __________.

Suppose without loss of generality that $a_1 \neq 0$. Then

(*) $\vec{u}_1=$

Let $\mathcal{H}=\{ \quad \}$. Then

$\vec{u}_1 \in$ ______. by ___, and

$\{ \quad \} \subset \text{Span}(\quad)$.

$V = \text{Span}(\{ \qquad\qquad\qquad\qquad \}) \subset \text{Span}(\qquad) \subset$ ---
and

$V = \text{Span}(\qquad\qquad), \quad \mathcal{H} \subset \mathcal{G},$

$|\mathcal{H}| =$ and

The Theorem is true for ----. The Theorem follows by mathematical induction. $\square$.

<u>Corollary 1.</u> Let V be a vector space with a finite basis. Then every basis for V has

-----------.

<u>PROOF</u> Suppose V has a finite basis $\mathcal{B}$ & $|\mathcal{B}| = n$.

<u>Case 1</u> $n = 0$. Then $V = \{ \quad \}$ and ----------.

<u>Case 2</u> $n \geq 1$. Let $\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \dots, \vec{b}_n\}$, ~~let~~

~~$\mathcal{B}' = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m\}$ be another basis.~~

~~Then $\mathcal{B}$ is ---------- and Span~~ .

~~so~~ So $\text{Span}(\mathcal{B}) = V$.

~~By Theorem 1.10 (Replacement Theorem) any linearly independent subset of V has at most n vectors.~~

Let $\mathcal{B}'$ be a basis for V. Then $\mathcal{B}'$ is linearly independent. If $\mathcal{B}'$ is infinite or has more than n vectors we can select a subset $\mathcal{C}$ of exactly $n+1$ elements. $\mathcal{C} \subset \mathcal{B}'$.

$\mathcal{C}$ is linearly independent since $\mathcal{B}'$ is. By Theorem 1.10 (Replacement Thm). $|\mathcal{C}| = n+1$ ----.

since

This is a __________.

Hence $\mathcal{B}'$ is ______ and $|\mathcal{B}'|$ ______.

Reversing the ________ and

we have $|\mathcal{B}|$

and therefore .

__Definition.__ A vector space V is __finite dimensional__ if ______________.

In this case the __dimension of V__ :=

and write __________.

If ____________________. then

we say that V is __infinite dimensional__.

__Examples__

(1) $\dim \{\vec{0}\} =$

(2) $\dim F^n =$

(3) $\dim P_n(F) =$

(4) $P(F)$ is

(5) $\dim M_{m\times n}(F) =$

(6) Let $\mathbb{C}$ be the set of complex nos (field).

$\mathbb{C}$ is a vector space over $\mathbb{C}$.

A basis is ______ & $\dim_{\mathbb{C}} \mathbb{C} =$

(7) $\mathbb{C} = \{x+iy : x, y \in \mathbb{R}\}$ is a vector space over $\mathbb{R}$.
A basis is
and $\dim_{\mathbb{R}} \mathbb{C} =$

Corollary 2. Suppose V is a finite dimensional vector space over F and $\dim V = n$.

(a) Let $\mathcal{G}$ be finite and suppose $\mathrm{Span}(\mathcal{G}) = V$.
Then $|\mathcal{G}|$ _____.
If ______ then $\mathcal{G}$ is a basis for V.

(b) Let
$\mathcal{L}$ be a linearly independent subset of V.
Then $|\mathcal{L}|$ _____.

If ________ then $\mathcal{L}$ is a basis for V.

(c) Suppose $\mathcal{L} \subset V$ is linearly independent.
Then
$\mathcal{L}$ ________ to form a basis for V.

PROOF: Let $\mathcal{B}$ be a basis for V. ~~and~~ Then ~~suppose~~ $|\mathcal{B}| =$ ___

(a) Suppose $\mathcal{G} \subset V$, $\mathcal{G}$ is finite & $\mathrm{Span}(\mathcal{G}) = V$.
By Theorem 1.10
_______.
since $|\mathcal{B}| =$ ___ and $\mathcal{B}$ is ______.

Now suppose $|\mathcal{G}| = n$.
By Theorem 1.9, some subset $\mathcal{G}' \subset$ _ _ _
is a _ _ _ _ _ _. But
$|\mathcal{G}'| =$ _ _ since _ _ _ _ _ _ _.

This implies _ _ _ _ since _ _ _ _ _ _.
Hence $\mathcal{G}$ is a basis for V.

(b) Suppose $\mathcal{L} \subset V$ and $\mathcal{L}$ is linearly independent.
Let $\mathcal{L}'$ be any finite subset of $\mathcal{L}$.
Then $\mathcal{L}'$ is _ _ _ _ _ and
by Theorem 1.10
$|\mathcal{L}'|$
It follows that $\mathcal{L}$ is _ _ _ and $|\mathcal{L}|$ _ _ _.

Now suppose $|\mathcal{L}| = n$. By Theorem 1.10
there is a subset $\mathcal{H}$ of $\mathcal{B}$ (since $\mathcal{B}$ _ _ _ _ _)
containing exactly _ _ _ _ vectors such that
$\mathcal{L} \cup \mathcal{H} =$ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _.
So $\mathcal{L}$ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _
and $\mathcal{L}$ is a basis for V.

(c) Suppose $\mathcal{L} \subseteq V$ & $\mathcal{L}$ is linearly independent.
Then by (b),

$$|\mathcal{L}| = m \text{ \_\_\_\_ }.$$

Since ______ and $|\mathcal{B}| =$ ____.
by Theorem 1.10 there is a subset $\mathcal{H}$ of ____.
Such that $|\mathcal{H}| =$ ___
and ______ spans V.

$$|\mathcal{L} \cup H| \leq \text{ \_\_\_\_\_\_\_ }.$$

By part (a)

_________.

and

$$|\mathcal{L} \cup H| = \text{ \_\_\_ } \text{ and}$$

______ is a basis for V by part (a).
Hence $\mathcal{L}$ can be extended ~~to a basis~~ the basis _____. $\square$

<u>Example</u>
Let $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 3 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \vec{v}_4 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \vec{v}_5 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \in \mathbb{R}^4.$

(i) Determine whether the vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ span $\mathbb{R}^4$
(ii) Determine whether the vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4, \vec{v}_5$ are linearly independent.

Theorem 1.11 Let W be a subspace of a finite dimensional vector space V. Then W is _ _ _ _ _ _ _ and

$$\dim W \_ \_ \_ \_ .$$

~~[illegible]~~

If $\dim W$ _ _ _ _ . Then _ _ _ _ .

PROOF: Let $\dim V = n$.

Case 1: $W = \{\vec{0}\}$.

Case 2 $W \neq \{\vec{0}\}$. Then W contains a vector $\vec{v} \neq$ _ _ _ .
Then $\{\vec{v}\}$ is a _ _ _ _ _ _ _ . subset of W.
Any linearly independent subset of W is linearly independent in _ _ and has _ _ _ _ _ by Cor. 2(b). Therefore we may choose a maximal linearly independent set $\mathcal{L}$ such that

$$\{\vec{v}\} \subset \mathcal{L} = \{\vec{v}_1, \ldots, \vec{v}_k\} \subset W$$

We will show that $\mathcal{L}$ is _ _ _ _ _ _.
By Cor. 2(b),

$$|\mathcal{L}| = k \text{ \_ \_ \_.}$$

We claim that $Span(\mathcal{L}) = W$.
Suppose by way of contradiction that

$$Span(\mathcal{L}) \subsetneq W,$$

ie there is _ _ _ _ such that _ _ _ _.
Therefore _ _ _ _ $\mathcal{L}$ and
_ _ _ _ _ _ is linearly independent by
Theorem 1.7. But

$$|\qquad| > \text{\_ \_ \_}$$

which contradicts the _ _ _ _ _ of $\mathcal{L}$.
Hence $Span(\mathcal{L}) = W$ and $\mathcal{L}$ is _ _ _ _ _.
Therefore $\dim(W)$ = _ _ _ _ _ _ _.

Now suppose $\dim W = \dim V = n$.
Let $\mathcal{D}$ be a basis for W. Then $|\mathcal{D}| = n$
and $\mathcal{D}$ is linearly independent. By Cor. 2(b)
$\mathcal{D}$ is a basis for V since $|\mathcal{D}| = |V| = n$ & $\mathcal{D}$
is linearly independent.
Since $\mathcal{D}$ is a basis for W, $Span(\mathcal{D}) = W$.
Since $\mathcal{D}$ is a basis for V, _ _ _ _ _,
and _ _ _ $\square$

Example Let

$$W = \left\{ \vec{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} \in F^5 : a_1 + a_3 + a_5 = 0,\ a_2 = a_4 \right\}$$

where F is a field. Then W is a subspace of F^5.
Show that

$$\mathcal{B} = \left\{ \vec{w}_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \ \vec{w}_2 = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \ \vec{w}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

is a basis for W. First

$\vec{w}_1 \in W$ since

$\vec{w}_2 \in W$ since

$\vec{w}_3 \in W$ since

and so $\mathcal{B} \subset$ ___.

Suppose $x_1 \vec{w}_1 + x_2 \vec{w}_2 + x_3 \vec{w}_3 = \vec{0}$ where $x_1, x_2, x_3 \in F$.

Now suppose $\vec{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} \in W$.

Then

$$\vec{a} = \begin{bmatrix} \\ \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} =$$

Hence $W =$

and thus $\mathcal{B}$ is a _ _ _ _ _,

and $\dim W =$ _ _ _.