MAS 4105 – EXAM 2 – FALL 2014

Friday, October 31.

NAME: Solution

Instructions: all should be written in a proper and coherent manner, and in a way any student in the class can follow your work. Show all necessary working and reasoning. When giving proofs your reasoning should be clear.

TOTAL 78 points

1. [8 points]

Complete:

(a) Definition: Let $V, W$ be vector spaces over a field $F$. A function $T: V \longrightarrow W$ is called a linear transformation if

(i) $T(\vec{v_1}+\vec{v_2}) = T(\vec{v_1})+T(\vec{v_2})$ for all $\vec{v_1}, \vec{v_2} \in V$, and

(ii) $T(c\vec{v}) = cT(\vec{v})$ for all $\vec{v} \in V$ & $c \in F$.

(b) Definition. Let $T: V \to W$ be linear where $V, W$ are vector spaces. The Null space of $T$ (or kernel of $T$) is defined by

$$N(T) := \{\vec{v} \in V : T(\vec{v}) = \vec{0}\}$$

The Range of $T$ (or image of $T$) is defined by

$$R(T) := \{T(\vec{v}) : \vec{v} \in V\}.$$

(c) <u>The Dimension Theorem</u> Let $V, W$ be vector spaces over a field $F$, and suppose $T: V \to W$ is linear. If $V$ is finite dimensional, then $R(T)$ is finite dimensional, and

$$\dim N(T) + \dim R(T) = \dim V$$

(d) <u>Definition</u> Suppose $\mathcal{B} = [\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n]$, $\mathcal{C} = [\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_m]$ are ordered bases of the vectorspaces $V, W$ respectively, and suppose $T: V \to W$ is linear. Then the matrix of $T$ with respect to (or relative to) the bases $\mathcal{B}$ & $\mathcal{C}$ is denoted and defined by

$$[T]_{\mathcal{B}}^{\mathcal{C}} = \left[ \begin{array}{c|c|c|c} [T(\vec{v}_1)]_{\mathcal{C}} & [T(\vec{v}_2)]_{\mathcal{C}} & \cdots & [T(\vec{v}_n)]_{\mathcal{C}} \end{array} \right]$$

<u>Note:</u> $[T]_{\mathcal{B}}^{\mathcal{C}}$ is an $m \times n$ matrix.

(e) <u>Theorem 2.14</u> Let $V, W$ be finite dimensional vector spaces with ordered bases $\mathcal{B}, \mathcal{C}$ respectively & suppose $T: V \to W$ is linear. Then

$$[T(\vec{v})]_{\mathcal{C}} = [T]_{\mathcal{B}}^{\mathcal{C}} [\vec{v}]_{\mathcal{B}}$$

for all $\vec{v} \in V$.

(f) Definition Let $\mathcal{B}, \mathcal{C}$ be ordered bases of a finite dimensional vector space V. The change of basis matrix (or the change of coordinate matrix) from $\mathcal{B}$ to $\mathcal{C}$ is the matrix

$$Q = [I_V]_{\mathcal{B}}^{\mathcal{C}}$$

(g) Theorem 2.22 Let $\mathcal{B}, \mathcal{C}$ be ordered bases of a finite dimensional vector space V. Let

$$Q = [I_V]_{\mathcal{B}}^{\mathcal{C}}$$

which is the change of basis matrix from $\mathcal{B}$ to $\mathcal{C}$. Then

$$[\vec{v}]_{\mathcal{C}} = Q\,[\vec{v}]_{\mathcal{B}},$$

for all $\vec{v} \in V$. Also $Q$ is an ______ matrix.

(h) Theorem

Diagram

$$\begin{tikzcd} {[\vec{v}]_{\mathcal{B}}} \arrow[r, "{[T]_{\mathcal{B}}}"] \arrow[d, "Q"] & {[T(\vec{v})]_{\mathcal{B}}} \\ {[\vec{v}]_{\mathcal{C}}} \arrow[r, "{[T]_{\mathcal{C}}}"] & {[T(\vec{v})]_{\mathcal{C}}} \arrow[u, "Q^{-1}"] \end{tikzcd}$$

Let $\mathcal{B}, \mathcal{C}$ be ordered bases of a finite dimensional vector space V. Suppose $T: V \to V$ is linear & $Q$ is the change of basis matrix from $\mathcal{B}$ to $\mathcal{C}$. Then

$$[T]_{\mathcal{B}} = Q^{-1}[T]_{\mathcal{C}}\,Q$$

2. [$3 \times 10 = 30$ points]

(a) Let $T: M_{2\times 2}(F) \to M_{2\times 2}(F)$ by

$$T\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} 2a_{11}-a_{12} & a_{11}+2a_{12} \\ 0 & 0 \end{pmatrix}$$

Prove that $T$ is linear

(b) You are given that

$T: M_{2\times 2}(\mathbb{R}) \to P_2(\mathbb{R})$ by

$$T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = (a+b) + 2dx + bx^2$$

is linear

(i) Find $[T]_{\beta}^{\gamma}$ where

$$\beta = \left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right],$$

$$\gamma = [1, x, x^2].$$

(ii) Find $\dim N(T)$.

(c) You are given that $\beta = [(1,2), (1,3)]$ is an ordered basis of $\mathbb{R}^2$. Suppose $\vec{v} \in \mathbb{R}^2$ & $[\vec{v}]_\beta = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$.

(i) Find $\vec{v}$.

(ii) You are given that $\gamma = [1+x, 1-x]$ is an ordered basis of $P_1(\mathbb{R})$. Suppose

$U: \mathbb{R}^2 \to P_1(\mathbb{R})$ is linear, and

$$\left[U\right]_{\mathcal{B}}^{\mathcal{C}} = \begin{bmatrix} -1 & 1 \\ -3 & 2 \end{bmatrix},$$

where $\mathcal{B}$ is given above.
Find $U(\vec{v})$ where $\vec{v}$ is given in (i).

(iii) Determine whether the transformation $U$ in (ii) is invertible.

3. Do THREE parts. [$3 \times 10 = 30$ points]

(a) Let $V$, $W$ be vector spaces & suppose $T: V \to W$ is linear. Prove that $N(T)$ is a subspace of $V$.

(b) Suppose $V$, $W$ are vector spaces and $T: V \to W$ is an invertible linear transformation. Prove that $T^{-1}: W \to V$ is linear.

(c) Let $V$, $W$ be finite dimensional vector spaces. Suppose $T: V \to W$ is linear and $\dim V > \dim W$. Prove that $T$ is not one-to-one.

(d) Which of the following vector spaces over $\mathbb{R}$ are isomorphic giving reasons?

$$\mathbb{R}^3, \mathbb{R}^4, P_2(\mathbb{R}), M_{2\times 2}(\mathbb{R}), \text{ and}$$

$$W = \{\vec{x} \in \mathbb{R}^5 : x_1 + x_2 + x_3 + x_4 + x_5 = 0\}.$$

4. $\boxed{5 \times 2 = 10 \text{ pts}}$

Let $T: P_1(\mathbb{R}) \to P_1(\mathbb{R})$ by

$$T(a+bx) = -(a+6b) + (a+4b)x.$$

You are given that $T$ is linear.

Let $\mathcal{B} = [2-x, \; 3-x]$, $\mathcal{C} = [1, x]$.

(a) Find the change of coordinate matrix from $\mathcal{B}$ to $\mathcal{C}$.

(b) Find the change of coordinate matrix from $\mathcal{C}$ to $\mathcal{B}$.

(c) Find $[T]_{\mathcal{B}}$

(d) Find $[T]_{\mathcal{C}}$

(e) Find an invertible matrix $Q$ such that

$$[T]_{\mathcal{B}} = Q^{-1}[T]_{\mathcal{C}}\, Q.$$

2.(a) $T$ is clearly well-defined.

Let $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, $B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \in M_{2\times 2}(F)$,

and $c \in F$. Then

$$A + B = \begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \end{pmatrix}$$

$$T(A+B) = \begin{pmatrix} 2(a_{11}+b_{11}) - (a_{12}+b_{12}) & a_{11}+b_{11}+2(a_{12}+b_{12}) \\ 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} (2a_{11}-a_{12}) + (2b_{11}-b_{12}) & (a_{11}+2a_{12}) + (b_{11}+2b_{12}) \\ 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 2a_{11}-a_{12} & a_{11}+2a_{12} \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 2b_{11}-b_{12} & b_{11}+2b_{12} \\ 0 & 0 \end{pmatrix}$$

$$= T(A) + T(B).$$

$$cA = \begin{pmatrix} ca_{11} & ca_{12} \\ ca_{21} & ca_{22} \end{pmatrix}.$$

$$T(cA) = \begin{pmatrix} 2ca_{11} - ca_{12} & ca_{11} + 2ca_{12} \\ 0 & 0 \end{pmatrix}$$

$$= c\begin{pmatrix} 2a_{11} - a_{12} & a_{11} + 2a_{12} \\ 0 & 0 \end{pmatrix} = cT(A)$$

Hence $T$ is linear.

(b) (i) $T\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = 1 = 1 + 0\cdot x + 0\cdot x^2.$

$T\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = 1 + x^2 = 1 + 0\cdot x + 1\cdot x^2.$

$T\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = 0 = 0 + 0\cdot x + 0\cdot x^2$

$T\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = 2x = 0\cdot 1 + 2\cdot x + 0\cdot x^2.$

$\therefore\ [T]_\beta^\gamma = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \end{bmatrix}$

(ii) $T$ is clearly onto since

$T\begin{pmatrix} \alpha-\beta & \beta \\ 0 & \gamma/2 \end{pmatrix} = \alpha + \gamma x + \beta x^2$ for $\alpha, \beta, \gamma \in \mathbb{R}$

Hence,

$$\dim N(T) = \dim M_{2\times 2}(\mathbb{R}) - \dim R(T)$$
$$= 4 - \dim P_2(\mathbb{R}) = 4 - 3 = 1,$$

(by the Dim. Thm.)

(c) (i) $\vec{v} = 3(1,2) + 4(1,3) = (7, 6+12) = (7, 18)$

(ii) $[U(\vec{v})]_{\mathcal{C}} = [U]_{\mathcal{B}}^{\mathcal{C}} [\vec{v}]_{\mathcal{B}}$

$$= \begin{pmatrix} -1 & 1 \\ -3 & 2 \end{pmatrix} \begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

So

$$U(\vec{v}) = 1\cdot(1+x) + (-1)(1-x) = 2x.$$

(iii) $\det [U]_{\mathcal{B}}^{\mathcal{C}} = -2+3 = 1 \neq 0.$

So $[U]_{\mathcal{B}}^{\mathcal{C}}$ is invertible & hence $U$ is invertible.

3.

(a) $N(T) = \{\vec{v} \in V : T(\vec{v}) = \vec{0}\} \subset V.$

Since $T$ is linear $T(\vec{0}) = \vec{0}$ & $\vec{0} \in N(T)$.

Let $\vec{v_1}, \vec{v_2} \in N(T)$ & $c \in F$.

i.e. $T(\vec{v_1}) = T(\vec{v_2}) = \vec{0}$.

$$T(\vec{v_1} + \vec{v_2}) = T(\vec{v_1}) + T(\vec{v_2}) \quad \text{(since } T \text{ is linear)}$$
$$= \vec{0} + \vec{0}$$
$$= \vec{0}$$

& $\vec{v_1} + \vec{v_2} \in N(T)$.

$$T(c\vec{v_1}) = cT(\vec{v_1}) \quad \text{(since } T \text{ is linear)}$$
$$= c\,\vec{0} = \vec{0},$$

& $c\vec{v_1} \in N(T)$. $N(T)$ is closed under addition & scalar multiplication & hence is a subspace of $V$.

(b) Suppose $T: V \to W$ is linear, & invertible.

Since $T$ is invertible, $T^{-1}$ exists & $T^{-1}: W \to V$.

We have $T \cdot T^{-1} = I_W$, $T^{-1}T = I_V$.

Let $\vec{w_1}, \vec{w_2} \in W$ & $c \in F$.

Let $\vec{v_1} = T^{-1}(\vec{w_1})$, $\vec{v_2} = T^{-1}(\vec{w_2})$.

Then

$$T(\vec{v_1}) = TT^{-1}(\vec{w_1}) = \vec{w_1},$$

$$\& \quad T(\vec{v_2}) = TT^{-1}(\vec{w_2}) = \vec{w_2}.$$

$$T(\vec{v_1} + \vec{v_2}) = T(\vec{v_1}) + T(\vec{v_2}) \quad \text{(since } T \text{ is linear)}$$
$$= \vec{w_1} + \vec{w_2}.$$

$$\text{So } T^{-1}(\vec{w_1} + \vec{w_2}) = T^{-1}T(\vec{v_1} + \vec{v_2})$$
$$= \vec{v_1} + \vec{v_2}$$
$$= T^{-1}(\vec{w_1}) + T^{-1}(\vec{w_2}).$$

$$T(c\vec{v_1}) = c\,T(\vec{v_1}) \quad \text{(since } T \text{ is linear)}$$
$$= c\,\vec{w_1}.$$

$$\text{So } T^{-1}(c\vec{w_1}) = T^{-1}T(c\vec{v_1}) = c\vec{v_1}$$
$$= c\,\vec{v_1}$$
$$= c\,T^{-1}(\vec{w_1}).$$

Hence $T^{-1}$ is linear. $\square$

(c) Suppose $T: V \to W$ is linear, $V, W$ finite dimensional & $\dim V > \dim W$.

$R(T)$ is a subspace of $W$ so

$\dim R(T) \leq \dim W < \dim V$.

By Dimension Theorem

$$\dim N(T) = \dim V - \dim R(T) > 0$$

since $\dim R(T) < \dim V$.

4.

(a) The change of basis matrix from $\mathcal{B}$ to $\mathcal{C}$ is

$$Q = [I_V]_{\mathcal{B}}^{\mathcal{C}} \quad \text{where} \quad V = P_1(\mathbb{R}).$$

$$= \left[ [2-x]_{\mathcal{C}} \quad [3-x]_{\mathcal{C}} \right] = \begin{bmatrix} 2 & 3 \\ -1 & -1 \end{bmatrix}.$$

(b) The change of basis matrix from $\mathcal{C}$ to $\mathcal{B}$ is

$$Q^{-1} = \frac{1}{\det Q} \begin{bmatrix} -1 & -3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} -1 & -3 \\ 1 & 2 \end{bmatrix}$$

(c) $$[T]_{\mathcal{B}} = \left[ [T(2-x)]_{\mathcal{B}} \quad [T(3-x)]_{\mathcal{B}} \right]$$

$$T(2-x) = -(2-6) + (2-4)x$$
$$= 4 - 2x = 2(2-x) + 0\cdot(3-x).$$
$$T(3-x) = -(3-6) + (3-4)x$$
$$= 3 - x = 0\cdot(2-x) + \cancel{3}1\cdot(3-x), \ \&$$

so
$$[T]_{\mathcal{B}} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}.$$

(d) $[T]_{\mathcal{C}} = \left[ [T(1)]_{\mathcal{C}} \quad [T(x)]_{\mathcal{C}} \right].$

$T(1) = -1 + x$

$T(x) = -6 + 4x$ &

$$[T]_{\mathcal{C}} = \begin{bmatrix} -1 & -6 \\ 1 & 4 \end{bmatrix}.$$

(e) Let $Q = \begin{bmatrix} 2 & 3 \\ -1 & -1 \end{bmatrix}$. Then by Thm 2.23

$$[T]_{\mathcal{B}} = Q^{-1}[T]_{\mathcal{C}}\,Q,$$

since $Q$ is the change of basis matrix from $\mathcal{B}$ to $\mathcal{C}$.

Check:

$$Q^{-1}[T]_{\mathcal{C}}Q = \begin{pmatrix} -1 & -3 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} -1 & -6 \\ 1 & 4 \end{pmatrix}\begin{pmatrix} 2 & 3 \\ -1 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} -2 & -6 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 2 & 3 \\ -1 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} = [T]_{\mathcal{B}} \text{ as hoped.}$$