1.4 Linear Combinations & Systems of Linear Equations

Defn

A *linear equation* in the variables $x_1, x_2, \dots, x_n$ has the form

$$a_1 x_1 + a_2 x_2 + \dots + a_n x_n = b$$

where $a_1, a_2, \dots, a_n, b$ are real constants.

Two linear systems are *equivalent* if they have the same solution set.

SOLVING A SYSTEM OF LINEAR EQUATIONS

Example Solve the system

$$\begin{aligned} 3x_2 - 6x_3 + 6x_4 + 4x_5 &= -5 \\ 3x_1 - 7x_2 + 8x_3 - 5x_4 + 8x_5 &= 9 \\ 3x_1 - 9x_2 + 12x_3 - 9x_4 + 6x_5 &= 15 \end{aligned}$$

$\longleftrightarrow$ (equivalent ie same soln set)

$$\begin{aligned} 3x_2 - 6x_3 + 6x_4 + 4x_5 &= -5 \\ 3x_1 - 7x_2 + 8x_3 - 5x_4 + 8x_5 &= 9 \\ x_1 - 3x_2 + 4x_3 - 3x_4 + 2x_5 &= 5 \end{aligned} \qquad (\tfrac{1}{3}R_3)$$

$(R_1 \Leftrightarrow R_2)$

$\longleftrightarrow$

$$\begin{aligned} x_1 - 3x_2 + 4x_3 - 3x_4 + 2x_5 &= 5 \\ 3x_1 - 7x_2 + 8x_3 - 5x_4 + 8x_5 &= 9 \\ 3x_2 - 6x_3 + 6x_4 + 4x_5 &= -5 \end{aligned}$$

$(-3R_1 + R_2)$

$\longleftrightarrow$

$$\begin{aligned} x_1 - 3x_2 + 4x_3 - 3x_4 + 2x_5 &= 5 \\ 2x_2 - 4x_3 + 4x_4 + 2x_5 &= -6 \\ 3x_2 - 6x_3 + 6x_4 + 4x_5 &= -5 \end{aligned}$$

$\Longleftrightarrow$ $(\frac{1}{2}R_2)$

$$\begin{aligned} x_1 - 3x_2 + 4x_3 - 3x_4 + 2x_5 &= 5 \\ x_2 - 2x_3 + 2x_4 + x_5 &= -3 \\ 3x_2 - 6x_3 + 6x_4 + 4x_5 &= -5 \end{aligned}$$

$\Longleftrightarrow$ $(-3R_2 + R_3)$

$$\begin{aligned} 1x_1 - 3x_2 + 4x_3 - 3x_4 + 2x_5 &= 5 \\ 1x_2 - 2x_3 + 2x_4 + x_5 &= -3 \\ 1x_5 &= 4 \end{aligned}$$

The leading variables are x_1, x_2, x_5.
Eliminate variables x_1, x_2, x_5 that are not leading if possible.

$\Longleftrightarrow$ $(-R_3 + R_2, -2R_3 + R_1)$

$$\begin{aligned} x_1 - 3x_2 + 4x_3 - 3x_4 &= -3 \\ x_2 - 2x_3 + 2x_4 &= -7 \\ x_5 &= 4 \end{aligned}$$

$\Longleftrightarrow$ $(3R_2 + R_1)$

$$\begin{aligned} x_1 - 2x_3 + 3x_4 &= -24 \\ x_2 - 2x_3 + 2x_4 &= -7 \\ x_5 &= 4 \end{aligned}$$

$\Longleftrightarrow$

$$\boxed{\begin{aligned} x_1 &= 2x_3 - 3x_4 - 24 \\ x_2 &= 2x_3 - 2x_4 - 7 \\ x_5 &= 4 \end{aligned}}$$

where x_3, x_4 are free variables (infinitely many solutions).

Solution Set

$= \{(x_1, x_2, x_3, x_4, x_5) : x_5 = 4,\ x_4 = a,\ x_3 = b,$
$x_2 = 2b - 2a - 7,$
$x_1 = 2b - 3a - 24,$
& where $a, b \in \mathbb{R}\}$.

Operations that preserve the solution set of a linear system with coefficients from a field F.

(1) Interchange two equations $(R_i \Leftrightarrow R_j)$
(2) Multiply an equation by a nonzero constant $(cR_i,\ c \neq 0)$
(3) Add a multiple of one equation to another $(cR_i + R_j)$ [Only one equation changes ie Equation j]

Properties of Required Linear System
(Reduced Echelon Form)
(1) The leading variables have coefficient 1
(ie. first nonzero coefficient in each equation is 1).
(2) A leading variable only occurs in one equation.
(3) The leading variable of each equation has a larger subscript than the leading variables of preceding equations.

The process of obtaining Reduced Echelon Form is called Gauss-Jordan Elimination.

NOTE: If we obtain an equation of the form $0 = c$ where $c \neq 0$, then the system is inconsistent & has no solutions.