MAS 4105 – Quiz 1 – Fall 2014

Wednesday, Sept. 3

NAME: Solution

Instructions: All work should be written in a proper and coherent manner, and in a way that any student in the class can follow your work. Show all necessary working and reasoning. When giving proofs your reasoning should be clear.

TOTAL:

1. [4 + 1 = 5 pts]

(a) Label each statement as true or false. No reasoning is necessary.

(i) A vector space may have more than one zero vector. False

(ii) In any vector space, $a\vec{x} = a\vec{y}$ implies that $\vec{x} = \vec{y}$. False.

(iii) If f and g are polynomials of degree n, then $f+g$ is a polynomial of degree n. False

(iv) Two functions in $\mathcal{F}(S, F)$ are equal if and only if they have the same value at each element of S. True

(b) Complete:

Theorem 1.1 (Cancellation Law)

Let V be a vector space over F. If $\vec{u}, \vec{v}, \vec{w} \in V$ and $\vec{u} + \vec{w} = \vec{v} + \vec{w}$,

then $\vec{u} = \vec{v}$.

2. [3 pts]

Complete:

Definition. A vector space V over a field F is a set V of mathematical objects with two operations (addition and scalar multiplication) defined so that for each pair $\vec{u}, \vec{v}$ in V there is a unique element $\vec{u}+\vec{v}$ in V and for each vector $\vec{u}$ in V and each scalar c in F there exists a unique element $c\vec{u}$ in V such that the following conditions hold:

(VS1) For all $\vec{u}, \vec{v}$ in V, $\vec{u}+\vec{v} = \vec{v}+\vec{u}$.

(VS2) For all $\vec{u}, \vec{v}, \vec{w}$ in V, $\vec{u}+(\vec{v}+\vec{w}) = (\vec{u}+\vec{v})+\vec{w}$.

(VS3) There exists an element $\vec{0}$ in V such that
$\vec{u}+\vec{0} = \vec{u}$ for all $\vec{u}$ in V

(VS4) For each $\vec{u} \in V$ there exists a vector $\vec{v} \in V$ such that $\vec{u}+\vec{v} = \vec{0}$.

(VS5) For each $\vec{u} \in V$, $1\vec{u} = \vec{u}$

(VS6) For all $a, b \in F$ and $\vec{u} \in V$,
$(ab)\vec{u} = a(b\vec{u})$.

(VS7) For all $a \in F$ and $\vec{u}, \vec{v} \in V$,
$a(\vec{u}+\vec{v}) = a\vec{u}+a\vec{v}$

(VS8) For all $a, b \in F$ and $\vec{u}$ in V,
$(a+b)\vec{u} = a\vec{u}+b\vec{u}$

3. [3 + 6 = 9 pk]

(i) Complete

Theorem 1.2 Let V be a vector space over a field F. Then

(a) If $\vec{u} \in V$ then $0\vec{u} = \underline{\vec{0}}$.

(b) If $\vec{u} \in V$ and $a \in F$ then

$$(-a)\vec{u} = -(\underline{a\vec{u}}) = a(\underline{-\vec{u}}).$$

(c) If $a \in F$ then $a\vec{0} = \underline{\vec{0}}$.

(ii) PROVE Theorem 1.2 (c).

Let $a \in F$.

$$\vec{0} + \vec{0} = \vec{0} \quad \text{by (VS3)}.$$

So $a(\vec{0} + \vec{0}) = a\vec{0}$.

But $a(\vec{0} + \vec{0}) = a\vec{0} + a\vec{0}$ by (VS7).

So

$$a\vec{0} + a\vec{0} = a\vec{0}.$$

Also

$$\vec{0} + a\vec{0} = a\vec{0} + \vec{0} \quad \text{(by (VS1))}$$

$$= a\vec{0} \quad \text{(by (VS3))}.$$

Hence

$$a\vec{0} + a\vec{0} = \vec{0} + a\vec{0}, \text{ and}$$

$$a\vec{0} = \vec{0}$$

by Theorem 1.1 (Cancellation Law).

4. [3 pts]

Let $V = \{(a_1, a_2) : a_1, a_2 \in F\}$, where F is a field.
Define addition of elements of V coordinatewise, and for $c \in F$ and $(a_1, a_2) \in V$ define

$$c(a_1, a_2) = (a_1, 0).$$

Is V a vectorspace over F with these operations?
Justify your answer.

By definition

$$1(a_1, a_2) = (a_1, 0) \text{ for all } (a_1, a_2) \in V.$$

So $1(1,1) = (1, 0)$,

and $1(1,1) \neq (1,1)$

since $(1,0) \neq (1,1)$. Therefore V does not satisfy (VS5).
Hence V is not a vector space over F.

5. [Bonus 2 pts]

Let $V = \{(a_1, a_2) \in \mathbb{R}^2 : a_2 > 0\}$.
Define

$$(a_1, a_2) + (b_1, b_2) = (a_1 b_2 + a_2 b_1,\ a_2 b_2),$$

and $c(a_1, a_2) = (c a_1 a_2^{c-1},\ a_2^c)$, for $c \in \mathbb{R}$.

PROVE that V with these operations satisfies (VS3).

Let $(a_1, a_2) \in V$. Then $(0, 1) \in V$ since $1 > 0$; &

$$(a_1, a_2) + (0, 1) = (a_1 \cdot 1 + a_2 \cdot 0,\ a_2 \cdot 1) = (a_1, a_2).$$

So V satisfies (VS3) with $\vec{0} = (0, 1)$.