MAS 4105 – Quiz 2 – Fall 2014

Wednesday, Sept. 17

NAME: Solution

Instructions: All work should be written in a proper and coherent manner, and in a way that any student in the class can follow your work. Show all necessary working and reasoning. When giving proofs your reasoning should be clear.

TOTAL: 20 pts + 2 bonus pts.

1. [1+1+3= 5 pts]

(a) Complete Definition: Let V be a vector space over a field F. Let $W \subset V$ be nonempty. W is a subspace of V if W is a vector space over F with operations of addition & scalar multiplication as defined in V.

(b) Complete Theorem 1.3 Let V be a vector space over of a field F and $W \subset V$. W is a subspace of V if and only if the following 3 conditions hold.

(a) $\vec{0} \in W$,

(b) $\vec{u} + \vec{v} \in W$ whenever $\vec{u}, \vec{v} \in W$,

and (c) $c\vec{u} \in W$ whenever $\vec{u} \in W$ & $c \in F$.

(c) Let $W = \{(a, b, c, d) \in \mathbb{R}^4 : a+2b+3c+4d=0\}$.
Prove that W is a subspace of $\mathbb{R}^4$.

$\vec{0} = (0,0,0,0) \in W$ since $0+2\cdot0+3\cdot0+4\cdot0=0$.

Let $\vec{u} = (u_1, u_2, u_3, u_4)$, $\vec{v} = (v_1, v_2, v_3, v_4) \in W$ & $c \in \mathbb{R}$

Then $u_1+2u_2+3u_3+4u_4 = v_1+2v_2+3v_3+4v_4 = 0$.

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$\vec{u} + \vec{v} = (u_1 + v_1, u_2 + v_2, u_3 + v_3, u_4 + v_4) \in W$
since $(u_1 + v_1) + 2(u_2 + v_2) + 3(u_3 + v_3) + 4(u_4 + v_4)$
$= (u_1 + 2u_2 + 3u_3 + 4u_4) + (v_1 + 2v_2 + 3v_3 + 4v_4) = 0 + 0 = 0.$

$c\vec{u} = (cu_1, cu_2, cu_3, cu_4) \in W$
since $(cu_1) + 2(cu_2) + 3(cu_3) + 4(cu_4)$
$= c(u_1 + 2u_2 + 3u_3 + 4u_4) = c \cdot 0 = 0.$

So W is subspace of $\mathbb{R}^4$ by Theorem 1.3, since W is also clearly a subset of $\mathbb{R}^4$ & satisfies (i), (ii), (iii) of Theorem 1.3

2. [2+3 = 5 pts]

(a) Complete Definition: Let V be a vector space and suppose $\phi \neq S \subset V$. We say a vector $\vec{v} \in V$ is a linear combination of vectors in S if there are vectors $\vec{s}_1, \vec{s}_2, \ldots, \vec{s}_n \in S$ & scalars $c_1, c_2, \ldots, c_n$ such that

$$\vec{v} = c_1\vec{s}_1 + c_2\vec{s}_2 + \cdots + c_n\vec{s}_n$$

The span of S denoted by Span(S) is defined by

$$\text{Span}(S) = \{\vec{v} : \vec{v} \in V \text{ \& } \vec{v} \text{ is a linear combination of vectors in } S\}.$$

By convention we define $\text{Span}(\phi) = \{\vec{0}\}$.

(b) Let $\vec{v} = (1, -1, 1, 2)$, and $S = \{(2,0,2,1), (1,1,1,-1)\} \subset \mathbb{R}^4$. Determine whether $\vec{v}$ is in the Span(S).

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$\vec{v} = (1,-1,1,2) = 1(2,0,2,1) + (-1)(1,1,1,-1)$.
which is a linear combination of vectors in S.

So $\vec{v} \in Span(S)$.

3. [1+1+3=5 pts] Complete Definition:

(a) Let $S \subset V$ where V is a vector space.
S is linearly dependent if there are distinct
vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n \in$ S and
scalars $c_1, c_2, \ldots, c_n$ not all zero
such that

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n = \vec{0}$$

Otherwise we say S is linearly independent.

(b) The empty set is linearly dependent.
True or False? Explain.
FALSE. Any linearly dependent set must contain at least one vector.

(c) Determine whether the set
$\{x^3 + 2x^2, -x^2+3x+1, x^3-x^2+2x-1\}$
is linearly dependent or linearly independent
in $P_3(\mathbb{R})$.

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Consider the eqn

$$a(x^3+2x^2)+b(-x^2+3x+1)+c(x^3-x^2+2x-1)=0$$

where a, b, c are real scalars.

$$\Leftrightarrow \begin{cases} a \quad + \quad c = 0 \\ 2a - b - c = 0 \\ 3b + 2c = 0 \\ b - c = 0 \end{cases}$$

$-2R_1+R_2$, $R_2 \Leftrightarrow R_3$

$$\Leftrightarrow \begin{cases} a \quad + \quad c = 0 \\ -b - 3c = 0 \\ b - c = 0 \\ 3b + 2c = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} a \quad + \quad c \\ -b - 3c = 0 \\ -4c = 0 \\ -7c = 0 \end{cases}$$

This implies $a=b=c=0$ & hence the set is linearly independent.

4. [5 pts]

Do ONE part (a) or (b).

(a) Let W_1, W_2 be subspaces of a vector space V. Prove that W_1+W_2 is a subspace of V.

(b) Let V be a vector space of a field of characteristic not equal to two.
Let $\vec{u}, \vec{v}$ be distinct vectors in V.
Prove that $\{\vec{u}, \vec{v}\}$ is linearly independent if and only if $\{\vec{u}+\vec{v}, \vec{u}-\vec{v}\}$ is linearly independent.

(a) Suppose W_1, W_2 are subspaces of V.
We know $W_1+W_2 \subset V$ (since V is closed under addition).
$\vec{0} \in W_1+W_2$ since $\vec{0}=\vec{0}+\vec{0}$
and $\vec{0} \in W_1$ & W_2 since W_1, W_2 are subspaces of V.
Let $\vec{u}, \vec{v} \in W_1+W_2$ & $c \in F$ (a scalar).
Then $\vec{u} = \vec{u_1}+\vec{u_2}$, where $\vec{u_1} \in W_1$ & $\vec{u_2} \in W_2$ &
$\vec{v} = \vec{v_1}+\vec{v_2}$, where $\vec{v_1} \in W_1$ & $\vec{v_2} \in W_2$.
$\vec{u_1}+\vec{v_1} \in W_1$ since W_1 is a subspace & $\vec{u_1}, \vec{v_1} \in W_1$.
$\vec{u_2}+\vec{v_2} \in W_2$ since W_2 is a subspace & $\vec{u_2}, \vec{v_2} \in W_2$.
Therefore $\vec{u}+\vec{v} = (\vec{u_1}+\vec{u_2})+(\vec{v_1}+\vec{v_2}) = (\vec{u_1}+\vec{v_1})+(\vec{u_2}+\vec{v_2}) \in W_1+W_2$.
So W_1+W_2 is closed under addition.
$c\vec{u} = c(\vec{u_1}+\vec{u_2}) = (c\vec{u_1})+(c\vec{u_2}) \in W_1+W_2$
since $c\vec{u_1} \in W_1$ since W_1 is a subspace & $\vec{u_1} \in W_1$, &
$c\vec{u_2} \in W_2$ since W_2 is a subspace & $\vec{u_2} \in W_2$.
Hence W_1+W_2 is closed under scalar multiplication &
W_1+W_2 is a subspace of V by Theorem 1.3. □

(b) Let V be a vector space over a field F with $\text{char}(F) \neq 2$
ie $1+1 \neq 0$ in F.

($\Longrightarrow$) Suppose $\{\vec{u}, \vec{v}\}$ is linearly indept.
Now suppose
$$a(\vec{u}+\vec{v}) + b(\vec{u}-\vec{v}) = \vec{0}$$
where $a, b \in F$. Then
$$(a+b)\vec{u} + (a-b)\vec{v} = \vec{0}.$$
This implies
$$(a+b) = 0 \qquad (1)$$
$$\& \quad (a-b) = 0 \qquad (2)$$
since $\{\vec{u}, \vec{v}\}$ is linearly indept.
(1)+(2) implies $(1+1)a = 0$ & $a = 0$ since $1+1 \neq 0$.
(1) implies $b = 0$ & so $a = b = 0$ &
$\{\vec{u}+\vec{v}, \vec{u}-\vec{v}\}$ is linearly indept.

($\Longleftarrow$) Suppose $\{\vec{u}+\vec{v}, \vec{u}-\vec{v}\}$ is linearly indept.
Suppose
$$c\,\vec{u} + d\,\vec{v} = \vec{0}$$
where $c, d \in F$. So

$$(\vec{u}+\vec{v}) + (\vec{u}-\vec{v}) = (1+1)\vec{u} \;\&$$
$$\vec{u} = \tfrac{1}{2}(\vec{u}+\vec{v}) + \tfrac{1}{2}(\vec{u}-\vec{v})$$
where $2 = 1+1 \neq 0$.
Also $(\vec{u}+\vec{v}) - (\vec{u}-\vec{v}) = (1+1)\vec{v}$ &
$$\vec{v} = \tfrac{1}{2}(\vec{u}+\vec{v}) - \tfrac{1}{2}(\vec{u}-\vec{v}).$$

Hence

$$c\left(\tfrac{1}{2}(\vec{u}+\vec{v})+\tfrac{1}{2}(\vec{u}-\vec{v})\right)+d\left(\tfrac{1}{2}(\vec{u}+\vec{v})-\tfrac{1}{2}(\vec{u}-\vec{v})\right)=\vec{0}$$

$$\tfrac{1}{2}(c+d)(\vec{u}+\vec{v})+\tfrac{1}{2}(c-d)(\vec{u}-\vec{v})=\vec{0}.$$

This implies

$$\tfrac{1}{2}(c+d)=0 \quad \& \qquad (3)$$

$$\tfrac{1}{2}(c-d)=0. \qquad (4)$$

(3)+(4) gives $c=0$,

(3)−(4) gives $d=0$.

Then $c=d=0$ & $\{\vec{u},\vec{v}\}$ is linearly indept. $\square$

5. [BONUS 2 pts]

Prove that if $\{A_1, A_2, \ldots, A_k\}$ is a linearly independent subset of $M_{m\times n}(F)$ then $\{A_1^t, A_2^t, \ldots, A_k^t\}$ is also linearly independent.

Suppose $\{A_1, A_2, \ldots, A_n\}$ ~~are~~ linearly indept..

Suppose

$$c_1 A_1^t + c_2 A_2^t + \cdots + c_k A_k^t = 0$$

where $c_1, c_2, \ldots, c_k \in F$. Then

$$(c_1 A_1^t + c_2 A_2^t + \cdots + c_k A_k^t)^t = 0^t$$

$$(c_1 A_1^t)^t + (c_2 A_2^t)^t + \cdots + (c_k A_k^t)^t = 0$$

(by induction & property of transpose). So

$$c_1 (A_1^t)^t + c_2 (A_2^t)^t + \cdots + c_k (A_k^t)^t = 0,$$

So

$$c_1 A_1 + c_2 A_2 + \cdots + c_k A_k = 0.$$

This implies $c_1 = c_2 = \cdots = c_k = 0$ since $\{A_1, A_2, \ldots, A_k\}$ is linearly indept.