MAS 4105 — Quiz 3 - Fall 2014

Wednesday, ~~October~~ 8

NAME: SOLUTION

Instructions: All work should be written in a proper and coherent manner, and in a way that any student in the class can follow your work. Show all necessary working and reasoning. When giving proofs your reasoning should be clear.

TOTAL: 20 pts + 2 bonus pts

1. [1+2+1+1 = 5 pts]

(a) Complete Definition: Let V, W be vector spaces over a field F. A function $T: V \to W$ is called a linear transformation if

(i) $T(\vec{v_1} + \vec{v_2}) = T(\vec{v_1}) + T(\vec{v_2})$ for all $\vec{v_1}, \vec{v_2} \in V$, and

(ii) $T(c\vec{v}) = cT(\vec{v})$ for all $c \in F$ & all $\vec{v} \in V$.

(b) Complete Definition: Let V, W be vector spaces and let $T: V \to W$ be linear. The Null Space of T (or kernel of T) is defined by

$$N(T) := \{\vec{v} \in V : T(\vec{v}) = \vec{0}\}.$$

The range of T (or image of T) is defined by

$$R(T) := \{T(\vec{v}) : \vec{v} \in V\}.$$

(c) Complete Theorem 2.2. Let V, W be vector spaces and suppose $T: V \to W$ is linear. If $\mathcal{B} = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is a __basis__ for V, then
$$R(T) = \mathrm{Span}(\underline{T(\mathcal{B})}) = \mathrm{Span}(\{\underline{T(\vec{v}_1), T(\vec{v}_2), \ldots, T(\vec{v}_n)}\}).$$

(d) Complete The Dimension Theorem. Let V, W be vector spaces over a field F, and suppose $T: V \to W$ is linear. If V is __finite dimensional__, then $R(T)$ is __finite dimensional__, and
$$\dim N(T) \underline{+ \dim R(T) = \dim V}.$$

2. [5 pts]

Let $T: \mathbb{R}^3 \to \mathbb{R}^2$ by $T(a_1, a_2, a_3) = (a_1 + a_2, a_2 + 2a_3)$. Prove that T is a linear transformation.

T is clearly well-defined. Let $\vec{a} = (a_1, a_2, a_3)$, $\vec{b} = (b_1, b_2, b_3) \in \mathbb{R}^3$ and $c \in \mathbb{R}$. Then $\vec{a} + \vec{b} = (a_1 + b_1, a_2 + b_2, a_3 + b_3)$,

$$T(\vec{a} + \vec{b}) = (a_1 + b_1 + a_2 + b_2, \; a_2 + b_2 + 2(a_3 + b_3))$$
$$= (a_1 + a_2 + b_1 + b_2, \; a_2 + 2a_3 + b_2 + 2b_3)$$
$$= (a_1 + a_2, a_2 + 2a_3) + (b_1 + b_2, b_2 + 2b_3)$$
$$= T(\vec{a}) + T(\vec{b}).$$

Also, $c\vec{a} = (ca_1, ca_2, ca_3)$. So

$$T(c\vec{a}) = (ca_1 + ca_2, \; ca_2 + 2ca_3)$$
$$= c(a_1 + a_2, a_2 + 2a_3)$$
$$= c\,T(\vec{a}).$$

Hence T is a linear transformation.

3. [5 pts]

Prove Theorem 2.1(ii): Let V, W be vector spaces over a field F and suppose $T: V \longrightarrow W$ is linear. Then $N(T)$ is a subspace of V.

$N(T) = \{\vec{v} \in V : T(\vec{v}) = \vec{0}\} \subset V$.

Since T is linear $T(\vec{0}) = \vec{0}$ & $\vec{0} \in N(T)$.

Let $\vec{v_1}, \vec{v_2} \in N(T)$ & $c \in F$.

Then $T(\vec{v_1}) = T(\vec{v_2}) = \vec{0}$.

$$\begin{aligned} T(\vec{v_1} + \vec{v_2}) &= T(\vec{v_1}) + T(\vec{v_2}) \quad \text{(since } T \text{ is linear)} \\ &= \vec{0} + \vec{0} \\ &= \vec{0} \end{aligned}$$

and $\vec{v_1} + \vec{v_2} \in N(T)$. So $N(T)$ is closed under addition.

$$\begin{aligned} T(c\vec{v_1}) &= cT(\vec{v_1}) \quad \text{(since } T \text{ is linear)} \\ &= c\vec{0} = \vec{0}, \end{aligned}$$

and $c\vec{v_1} \in N(T)$. So $N(T)$ is closed under scalar multiplication.

Hence $N(T)$ is a subspace of V by Thm 1.3.

4. [5 pts]

You are given that $T: P_2(\mathbb{R}) \longrightarrow P_2(\mathbb{R})$ defined by $T(f(x)) = f(x) - xf'(x)$ is a linear transformation.

Find $\dim N(T)$ and $\dim R(T)$.

Let $f(x) = a_0 + a_1 x + a_2 x^2$.

Then $f'(x) = a_1 + 2a_2 x$

$$\begin{aligned} T(f) = f(x) - xf'(x) &= a_0 + a_1 x + a_2 x^2 - x(a_1 + 2a_2 x) \\ &= a_0 - a_2 x^2 = 0 \end{aligned}$$

iff $a_0 = a_2 = 0$. So $N(T) = \{a_1 x : a_1 \in \mathbb{R}\} = \operatorname{span}(\{x\})$.

$\{x\}$ is indept. & a basis for $N(T)$.
Hence $\dim N(T) = 1$.
By Dimension Theorem

$$\dim N(T) + \dim R(T) = \dim P_2(\mathbb{R}),$$
$$1 + \dim R(T) = 3,$$
$$\dim R(T) = 2.$$

5. [2 bonus pts]
Let V, W be finite dimensional vector spaces & suppose $T: V \to W$ is linear.
Prove that if $\dim V < \dim W$ then T can not be onto.

Suppose $\dim V < \dim W$.
By the Dimension Theorem

$$\dim R(T) + \dim N(T) = \dim V,$$
$$\dim R(T) = \dim V - \dim N(T)$$
$$\leq \dim V \quad \text{since } \dim N(T) \geq 0.$$

So $\dim R(T) \leq \dim V < \dim W$ &

$$\dim R(T) < \dim W.$$

Therefore $R(T) \neq W$ &
T is not <u>onto</u>.

ALTERNATIVE for Computing $\dim R(T)$ in Qu. 4

$\beta = \{1, x, x^2\}$ is a basis for $V = P_2(\mathbb{R})$.

$$T(1) = 1 - 0 = 1$$
$$T(x) = x - x = 0$$
$$T(x^2) = x^2 - x(2x) = -x^2.$$

So

$T(\beta) = \{1, 0, -x^2\}$ spans $R(T)$ by a Theorem (See Qu 1. (c)).

$$\text{Span}(T(\beta)) = \text{Span}(\{1, 0, -x^2\})$$
$$= \text{Span}(\{1, -x^2\}).$$

$\{1, -x^2\}$ is clearly linearly indept. & hence forms a basis for $R(T)$. So

$$\dim R(T) = 2$$

and by the Dimension Theorem

$$\dim N(T) = \dim P_2(\mathbb{R}) - \dim R(T)$$
$$= 3 - 2 = 1.$$