MAS 4105 — Quiz 4 — Fall 2014

Wednesday, ~~October 22~~

NAME: SOLUTION

Instructions: all work should be written in a proper and coherent manner, and in a way that any student in the class can follow your work. Show all necessary working and reasoning. When giving proofs your reasoning should be clear.

TOTAL: 20 pts + 2 bonus pts.

1. [2 + 1 + 1 + 1 = 5 pts]

(a) Complete Definition: Suppose $\mathcal{B} = [\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n]$ is an ordered basis of V and $\mathcal{C} = [\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_m]$ is an ordered basis of W and $T : V \longrightarrow W$ is linear. Then the matrix of the transformation T with respect (or relative) to the bases $\mathcal{B}$ and $\mathcal{C}$ is denoted and defined by

$$[T]_{\mathcal{B}}^{\mathcal{C}} = \left[\; [T(\vec{v}_1)]_{\mathcal{C}} \;\vdots\; [T(\vec{v}_2)]_{\mathcal{C}} \;\vdots\; \cdots \;\vdots\; [T(\vec{v}_n)]_{\mathcal{C}} \; \right]$$

Note: $[T]_{\mathcal{B}}^{\mathcal{C}}$ is a m × n matrix.

(b) Complete Definition: Let A be an $m \times n$ matrix, and B be a $n \times p$ matrix. Then the product AB is a $m \times p$ matrix defined by

$$(AB)_{ij} = \sum_{k=1}^{n} A_{ik} B_{kj}$$

for $1 \le i \le m$ and $1 \le j \le p$.

(c) Complete Theorem 2.14 Let V, W be finite dimensional vector spaces with ordered bases $\mathcal{B}, \mathcal{C}$ respectively. Let $T: V \longrightarrow W$ be linear. Then

$$\left[T(\vec{v}) \right]_{\mathcal{C}} = \left[T \right]_{\mathcal{B}}^{\mathcal{C}} \left[\vec{v} \right]_{\mathcal{B}}$$

for all $\vec{v} \in V$.

(d) Complete Theorem 2.15(4) If $T: F^n \longrightarrow F^m$ is linear then there exists a unique matrix $C \in M_{m \times n}(F)$ such that

$$T = L_C, \quad \text{namely } C = \left[T \right]_{\mathcal{B}}^{\mathcal{C}}.$$

where $\mathcal{B}$ is standard basis of F^n & $\mathcal{C}$ is standard basis of F^m.

2. [5 pts]

Define $L: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ by $L(\vec{x}) = A\vec{x}$ where $A = \begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix}$, so that L is linear. Let

$$\mathcal{B} = \left[\vec{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \ \vec{v}_2 = \begin{pmatrix} 3 \\ 4 \end{pmatrix} \right],$$

which is an ordered basis for $\mathbb{R}^2$. Find the matrix

$$B = \left[L \right]_{\mathcal{B}} = \left[L \right]_{\mathcal{B}}^{\mathcal{B}}.$$

$$L(\vec{v_1}) = \begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \end{pmatrix} = -2\vec{v_1}$$

$$[L(\vec{v_1})]_{\mathcal{B}} = \begin{bmatrix} -2 \\ 0 \end{bmatrix}.$$

$$L(\vec{v_2}) = \begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix}\begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 15 \\ 20 \end{pmatrix} = 5\vec{v_2}$$

$$[L(\vec{v_2})]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 5 \end{bmatrix} \quad \&$$

$$B = [L]_{\mathcal{B}} = \begin{bmatrix} -2 & 0 \\ 0 & 5 \end{bmatrix}$$

3. [2 + 3 = 5 pts]

(a) You are given that $\mathcal{B} = [1+x, 1-x]$ is an ordered basis for $P_1(\mathbb{R})$. Suppose $p(x) \in P_1(\mathbb{R})$ and

$$[p(x)]_{\mathcal{B}} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}. \quad \text{Find } p(x).$$

$$p(x) = 2(1+x) + 3(1-x) = 5 - x.$$

(b) You are given that $\mathcal{C} = [(2,3), (1,1)]$ is ~~and~~ an ordered basis for $\mathbb{R}^2$. Suppose $T: P_1(\mathbb{R}) \to \mathbb{R}^2$ is a linear transformation and

$$[T]_{\mathcal{B}}^{\mathcal{C}} = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}$$

where $\mathcal{B}$ is given (a). Find $T(p(x))$ where $p(x)$ is given in (a).

(p. 4)

$$\left[T(p(x))\right]_\gamma = [T]_\beta^\gamma \, [p(x)]_\beta \quad \text{(Thm 2.14)}$$

$$= \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 5 \\ 11 \end{pmatrix}. \quad \text{So}$$

$$T(p(x)) = 5(2,3) + 11(1,1)$$

$$= (21, 26).$$

4. [5 pts]

Prove Theorem 2.9: Let V, W, Z be vector spaces over F. Let $T: V \longrightarrow W$, and $U: W \longrightarrow Z$ be linear. Then $UT: V \longrightarrow Z$ defined by $UT(\vec{v}) = U(T(\vec{v}))$ is linear.

PROOF: Let $\vec{v} \in V$, then $T(\vec{v}) \in W$ & $U(T(\vec{v}))$ is defined & $U(T(\vec{v})) \in Z$. So $UT: V \longrightarrow Z$ is well-defined.

Let $\vec{v_1}, \vec{v_2} \in V$ & $c \in F$.

$$UT(\vec{v_1} + \vec{v_2}) = U(T(\vec{v_1} + \vec{v_2}))$$

$$= U(T(\vec{v_1}) + T(\vec{v_2})) \quad \text{(since } T \text{ is linear)}$$

$$= U(T(\vec{v_1})) + U(T(\vec{v_2})) \quad \text{(since } U \text{ is linear)}$$

$$= UT(\vec{v_1}) + UT(\vec{v_2}).$$

$$UT(c\vec{v_1}) = U(T(c\vec{v_1}))$$
$$= U(cT(\vec{v_1})) \quad \text{(since } T \text{ is linear)}$$
$$= c\,U(T(\vec{v_1})) \quad \text{(since } U \text{ is linear)}$$
$$= c\,UT(\vec{v_1}).$$

Hence UT is linear.

5. [2 bonus points].
Find linear transformations $U, T: F^2 \to F^2$ such that
$$UT = T_0 \quad \text{(the zero transformation)}$$
but
$$TU \neq T_0.$$

We construct U, T with $R(T) = N(U) = \text{Span}\left(\left\{\begin{bmatrix}1\\1\end{bmatrix}\right\}\right)$

Let $T: F^2 \to F^2$ by $T\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}x\\x\end{pmatrix} = \begin{pmatrix}1 & 0\\1 & 0\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}$.
Then T is linear since $T = L_A$, where $A = \begin{pmatrix}1 & 0\\1 & 0\end{pmatrix}$.

~~Also~~ Let $U: F^2 \to F^2$ by $U\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}x-y\\x-y\end{pmatrix} = \begin{pmatrix}1 & -1\\1 & -1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}$.

Then U is linear since $T = L_B$ where $B = \begin{pmatrix}1 & -1\\1 & -1\end{pmatrix}$.

Let $\begin{pmatrix}x\\y\end{pmatrix} \in F^2$. Then

$$U\left(T\begin{pmatrix}x\\y\end{pmatrix}\right) = U\begin{pmatrix}x\\x\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix} = \vec{0} \quad \text{for all } \begin{pmatrix}x\\y\end{pmatrix} \in F^2$$

So $UT = T_0$.