Group Number

MAS 4105 — GROUP PRACTICE EXAM (Quiz 5).

Tuesday, Oct 28, 2014.

Names: 1. Solution

2.

3.

4.

5.

Instructions: All work should be written in a proper & coherent manner, and in a way any student in the class can follow your work. Show all necessary working and reasoning. When giving proofs your reasoning should be clear.

TOTAL: 88 pts (To be rescaled to 20 pts).

1. [8 points]

Complete:

(a) Definition: Let V, W be vector spaces over a field F. A function $T: V \to W$ is called a linear transformation if (i) $T(\vec{v_1}+\vec{v_2}) = T(\vec{v_1}) + T(\vec{v_2})$, and

(ii) $T(c\vec{v_1}) = cT(\vec{v_1})$ for all $c \in F$ & $\vec{v_1}, \vec{v_2} \in V$.

(b) Definition Let $T: V \to W$ be linear where V, W are vector spaces. The Null space of T (or kernel of T) is defined by

$$N(T) := \{\vec{v} \in V : T(\vec{v}) = \vec{0}\}.$$

The range of T (or image of T) is defined by

$$R(T) := \{T(\vec{v}) : \vec{v} \in V\}$$

(c) The Dimension Theorem Let V, W be vector spaces over a field F, and suppose $T: V \longrightarrow W$ is linear.
If V is finite dimensional, then
$R(T)$ is finite dimensional, and

$$\dim N(T) + \dim R(T) = \dim V.$$

(d) Suppose $\mathcal{B} = [\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n]$ is an ordered basis of V and
$\mathcal{C} = [\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_m]$ is an ordered basis of W, and
$T: V \to W$ is linear. Then the matrix of T with respect to (or relative to) the bases $\mathcal{B}$ and $\mathcal{C}$ is denoted & defined by

$$[T]_{\mathcal{B}}^{\mathcal{C}} = \left[[T(\vec{v}_1)]_{\mathcal{C}} \;\vdots\; [T(\vec{v}_2)]_{\mathcal{C}} \;\vdots\; \cdots \;\vdots\; [T(\vec{v}_n)]_{\mathcal{C}} \right]$$

Note $[T]_{\mathcal{B}}^{\mathcal{C}}$ is a $m \times n$ matrix.

(e) Theorem 2.14 Let V, W be finite dimensional vector spaces with ordered bases $\mathcal{B}$, $\mathcal{C}$ respectively. Let $T: V \to W$ be linear. Then

$$[T(\vec{v})]_{\mathcal{C}} = [T]_{\mathcal{B}}^{\mathcal{C}} [\vec{v}]_{\mathcal{B}},$$

for all $\vec{v} \in V$.

(f) Definition Let $\mathcal{B}$, $\mathcal{C}$ be ordered bases of a finite dimensional vector space V. The change of basis matrix (or the change of coordinate matrix) from $\mathcal{B}$ to $\mathcal{C}$ is the matrix

$$Q = [I_V]_{\mathcal{B}}^{\mathcal{C}}.$$

(g) <u>Theorem 2.22</u> Let $\mathcal{B}, \mathcal{C}$ be ordered bases of a finite dimensional vector space V. Let $Q = [I_V]_{\mathcal{B}}^{\mathcal{C}}$

(which is the change of basis matrix from $\mathcal{B}$ to $\mathcal{C}$). Then Q is invertible, and for any $\vec{v} \in V$

$$[\vec{v}]_{\mathcal{C}} = Q \, [\vec{v}]_{\mathcal{B}}.$$

(h) <u>Theorem 2.23</u> Let $\mathcal{B}, \mathcal{C}$ be ordered bases of a finite dimensional vector space V. Suppose $T: V \to V$ is linear and Q is the change of basis matrix from $\mathcal{B}$ to $\mathcal{C}$. Then

$$[T]_{\mathcal{B}} = Q^{-1} [T]_{\mathcal{C}} \, Q$$

<u>Diagram</u>

$$\begin{tikzcd}
{[\vec{v}]_{\mathcal{B}}} \arrow[r, "\boxed{[T]_{\mathcal{B}}}"] \arrow[d, "\boxed{Q}"'] & {[T(\vec{v})]_{\mathcal{B}}} \\
{[\vec{v}]_{\mathcal{C}}} \arrow[r, "\boxed{[T]_{\mathcal{C}}}"] & {[T(\vec{v})]_{\mathcal{C}}} \arrow[u, "\boxed{Q^{-1}}"']
\end{tikzcd}$$

2. $[10 + (5+5) + (2+5+3) = 30\text{pts}]$

(a) Let $T: P_2(\mathbb{R}) \to P_3(\mathbb{R})$ by $T(f(x)) = xf(x) + f'(x)$. Prove T is linear.

(b) You are given that $T: M_{2\times 2}(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ by $T(A) = A^t$ is linear.

(i) Find $[T]_{\mathcal{B}}$ where

$$\mathcal{B} = \left[B_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, B_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, B_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, B_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right],$$

is the standard basis of $M_{2\times 2}(\mathbb{R})$.

(ii) Determine whether the transformation T is invertible

(c) You are given that $\mathcal{B} = [(1,2), (1,3)]$ is an ordered basis of $\mathbb{R}^2$. Suppose $\vec{v} \in \mathbb{R}^2$ & $[\vec{v}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$.

(i) Find $\vec{v}$

(ii) You are given that $\mathcal{G} = [1+x, 1-x]$ is an ordered basis for $P_1(\mathbb{R})$. Suppose $U: \mathbb{R}^2 \to P_1(\mathbb{R})$ is linear, &

$$[U]_{\mathcal{B}}^{\mathcal{G}} = \begin{bmatrix} -1 & 1 \\ -3 & 2 \end{bmatrix},$$

where $\mathcal{B}$ is given above.

Find $U(\vec{v})$ where $\vec{v}$ is given in (i).

(iii) Determine whether the transformation U in (ii) is invertible.

3. Do FOUR parts $[4 \times 10 = 40 \text{ pts}]$

(a) Suppose $T: V \to W$ is linear.
Prove $R(T)$ is a subspace of W.

(b) Suppose $T: V \to W$ is an invertible linear transformation.
Prove that $T^{-1}: W \to V$ is linear.

(c) Suppose $T: V \to W$ is linear &
$S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\} \subseteq V$. Prove that if $\{T(\vec{v}_1), T(\vec{v}_2), \ldots, T(\vec{v}_n)\}$ is a linearly independent subset of $R(T)$ then S is linearly independent subset of V.

(d) Let $T: V \to W$ be an invertible linear transformation.
Prove that if V is finite dimensional then W is finite dimensional & $\dim V = \dim W$.

(e) Which of the following vector spaces over $\mathbb{R}$ are isomorphic giving reasons?

$\mathbb{R}^3$, $\mathbb{R}^4$, $P_3(\mathbb{R})$, $M_{2\times 2}(\mathbb{R})$, $W = \{\vec{x} \in \mathbb{R}^4 : x_1 + 2x_2 + 3x_3 + 4x_4 = 0\}$

4. [$5 \times 2 = 10$ pts].

Let $T: P_1(\mathbb{R}) \longrightarrow P_1(\mathbb{R})$ by

$T(a + bx) = (a+2b) + (2a+b)x$.

You are given that T is linear.

Let

$$\mathcal{B} = [1+x, 1-x], \quad \mathcal{C} = [1, x].$$

(a) Find the change of coordinate matrix from $\mathcal{B}$ to $\mathcal{C}$.

(b) Find the change of coordinate matrix from $\mathcal{C}$ to $\mathcal{B}$

(c) Find $[T]_{\mathcal{B}}$

d) Find $[T]_{\mathcal{C}}$.

(e) Find an invertible matrix Q such that

$$[T]_{\mathcal{B}} = Q^{-1} [T]_{\mathcal{C}} Q.$$

2.

(a) Let $p(x) \in P_2(\mathbb{R})$. Then $\deg p \le 2$
so $\deg xp \le 3$ & $\deg p' \le 1$ &
so $\deg(xp(x) + p'(x)) \le 3$ & T is well-defined
since $T(p(x)) \in P_3(\mathbb{R})$.

Let $f(x), g(x) \in P_2(\mathbb{R})$ & $c \in \mathbb{R}$.
Then

$$T(f(x) + g(x)) = x(f(x) + g(x)) + \frac{d}{dx}(f(x) + g(x))$$
$$= xf(x) + xg(x) + f'(x) + g'(x)$$
$$= (xf(x) + f'(x)) + (xg(x) + g'(x))$$
$$= T(f(x)) + T(g(x)).$$

Also

$$T(cf(x)) = x(cf(x)) + \frac{d}{dx}(cf(x))$$
$$= c\,xf(x) + c\,f'(x)$$
$$= c\,(xf(x) + f'(x))$$
$$= c\,T(f(x)).$$

Hence T is linear.

2.

(b)(i) $T(B_1) = B_1^t = B_1 = 1\cdot B_1 + 0\cdot B_2 + 0\cdot B_3 + 0\cdot B_4$

$T(B_2) = B_2^t = B_3 = 0\cdot B_1 + 0\cdot B_2 + 1\cdot B_3 + 0\cdot B_4$

$T(B_3) = B_3^t = B_2 = 0\cdot B_1 + 1\cdot B_2 + 0\cdot B_3 + 0\cdot B_4$

$T(B_4) = B_4^t = B_4 = 0\cdot B_1 + 0\cdot B_2 + 0\cdot B_3 + 1\cdot B_4$. So

$$[T]_{\mathcal{B}} = \left[\, [T(B_1)]_{\mathcal{B}} \;\; [T(B_2)]_{\mathcal{B}} \;\; [T(B_3)]_{\mathcal{B}} \;\; [T(B_4)]_{\mathcal{B}} \,\right]$$

$$= \left[\, [B_1]_{\mathcal{B}} \;\; [B_3]_{\mathcal{B}} \;\; [B_2]_{\mathcal{B}} \;\; [B_4]_{\mathcal{B}} \,\right]$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

(ii) $T\cdot T(A) = T(T(A)) = T(A^t) = (A^t)^t = A.$

So $T\cdot T = I_{M_{2\times 2}(\mathbb{R})}$.

Hence T is invertible & $T^{-1} = T$.

2(c) (i) $[\vec{v}]_{\beta} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

$\therefore\ \vec{v} = 3(1,2) + 4(1,3)$
$= (7, 18).$

(ii) By Thm 2.4
$$[U(\vec{v})]_{\gamma} = [U]_{\beta}^{\gamma} [\vec{v}]_{\beta}$$
$$= \begin{pmatrix} -1 & 1 \\ -3 & 2 \end{pmatrix} \begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

So $U(\vec{v}) = 1(1+x) + (-1)(1-x)$
$= 2x.$

(iii). $[U]_{\beta}^{\gamma}$ is an invertible matrix

since it is 2×2 & $\det = -2+3 = 1 \neq 0$.
$\therefore$ the transformation U is invertible.

3.

(a) $R(T) = \{T(\vec{v}) : \vec{v} \in V\} \subset W$
since $T: V \to W$.
Since T is linear, $T(\vec{0}) = \vec{0}$ & $\vec{0} \in R(T)$.
Suppose $\vec{w_1}, \vec{w_2} \in R(T)$ & $c \in F$.
Then $T(\vec{v_1}) = \vec{w_1}$, $T(\vec{v_2}) = \vec{w_2}$ for some $\vec{v_1}, \vec{v_2} \in V$.

$\vec{w_1} + \vec{w_2} = T(\vec{v_1}) + T(\vec{v_2})$

$= T(\vec{v_1} + \vec{v_2})$ since T is linear.

Hence $\vec{w_1} + \vec{w_2} \in R(T)$ since $\vec{v_1} + \vec{v_2} \in V$ &
$T(\vec{v_1} + \vec{v_2}) = \vec{w_1} + \vec{w_2}$.

$c\vec{w_1} = c\,T(\vec{v_1}) = T(c\vec{v_1})$ since T is linear.

$c\vec{w_1} \in R(T)$ since $T(c\vec{v_1}) = c\vec{w_1}$ & $c\vec{v_1} \in V$.

Hence $R(T)$ is closed under addition & scalar multiplication, and so $R(T)$ is a subspace of W.

3(b) Suppose $T: V \to W$ is an invertible linear transformation.
Then $T^{-1}: W \to V$ &

$$TT^{-1} = I_W \quad \& \quad T^{-1}T = I_V.$$

Suppose $\vec{w_1}, \vec{w_2} \in W$ & $c \in F$.
Let $\vec{v_1} = T^{-1}(\vec{w_1})$, $\vec{v_2} = T^{-1}(\vec{w_2})$. Then

$$T(\vec{v_1}) = TT^{-1}(\vec{w_1}) = \vec{w_1},$$
$$T(\vec{v_2}) = TT^{-1}(\vec{w_2}) = \vec{w_2}.$$
$$\vec{w_1} + \vec{v_2} = T(\vec{v_1}) + T(\vec{v_2})$$
$$= T(\vec{v_1} + \vec{v_2}) \text{ since } T \text{ is linear.}$$

& $T^{-1}(\vec{w_1} + \vec{w_2}) = T^{-1}T(\vec{v_1} + \vec{v_2})$

$$= \vec{v_1} + \vec{v_2} = T^{-1}(\vec{w_1}) + T^{-1}(\vec{w_2}).$$

$$T(c\vec{v_1}) = cT(\vec{v_1}) = c\vec{w_1} \qquad \text{since } T \text{ is linear}$$

& $T^{-1}(c\vec{w_1}) = T^{-1}T(c\vec{v_1}) =$

$$= c\vec{v_1} = cT^{-1}(\vec{w_1}).$$

Hence we have

$$T^{-1}(\vec{w_1} + \vec{w_2}) = T^{-1}(\vec{w_1}) + T^{-1}(\vec{w_2}), \; \&$$
$$T^{-1}(c\vec{w_1}) = cT^{-1}(\vec{w_1}),$$

for all $\vec{w_1}, \vec{w_2} \in W$ & $c \in F$. Therefore T^{-1} is linear.

3(c) Suppose $\{T(\vec{v_1}), T(\vec{v_2}), \ldots, T(\vec{v_n})\} = T(S)$ is linearly independent.

Suppose

$$c_1 \vec{v_1} + c_2 \vec{v_2} + \cdots + c_n \vec{v_n} = \vec{0}$$

where $c_1, c_2, \ldots, c_n \in F$. Then

$T(c_1 \vec{v_1} + c_2 \vec{v_2} + \cdots + c_n \vec{v_n}) = T(\vec{0}) = \vec{0}$ since T is linear. But

$$T(c_1 \vec{v_1} + c_2 \vec{v_2} + \cdots + c_n \vec{v_n}) = c_1 T(\vec{v_1}) + c_2 T(\vec{v_2}) + \cdots + c_n T(\vec{v_n}),$$

&

$$c_1 T(\vec{v_1}) + c_2 T(\vec{v_2}) + \cdots + c_n T(\vec{v_n}) = \vec{0}.$$

This implies $c_1 = c_2 = \cdots = c_n = 0$ since $T(S)$ is linearly indept. Therefore the elements of S are distinct & S is linearly independent.

3.

(d) Let $T: V \longrightarrow W$ be an invertible linear transformation.
Suppose V is finite dimensional.
Let $\beta = \{\vec{v_1}, \vec{v_2}, \ldots, \vec{v_n}\}$ be a basis for V.
Then $T(\beta) = \{T(\vec{v_1}), T(\vec{v_2}), \ldots, T(\vec{v_n})\}$ spans $R(T)$.
~~Since~~ T is onto since T is invertible & $R(T) = W$.
Thus W has a finite spanning set & so as subset $T(\beta)$ is a basis & so W is finite dimensional.
By the Dimension Theorem

$$\dim N(T) + \dim R(T) = \dim V.$$

But $\dim N(T) = 0$ since T is invertible & one-to-one

$$\dim R(T) = \dim W \quad \text{since } R(T) = W.$$

Therefore $\dim V = \dim W$.

3 (e) $\dim \mathbb{R}^3 = 3$,

$\dim \mathbb{R}^4 = 4$,

$\dim P_3(\mathbb{R}) = 3+1 = 4$,

$\dim M_{2\times 2}(\mathbb{R}) = 2\cdot 2 = 4$.

$\vec{x} \in W$ iff $x_1 = -2x_2 - 3x_3 - 4x_4$ where x_2, x_3, x_4 are arbitrary

ie $\vec{x} = (-2x_2 - 3x_3 - 4x_4, x_2, x_3, x_4)$

$= x_2(-2,1,0,0) + x_3(-3,0,1,0) + x_4(-4,0,0,1)$

where $x_2, x_3, x_4 \in \mathbb{R}$. So

$\mathcal{B} = \{(-2,1,0,0), (-3,0,1,0), (-4,0,0,1)\}$ spans W &

is clearly linearly indept (since lin. comb. $= \vec{0}$ forces

$x_2 = x_3 = x_4 = 0$). So $\mathcal{B}$ is a basis for W & $\dim W = 3$.

Finite dim. vector spaces over (over $F = \mathbb{R}$) are isomorphic

iff they have the same dimension. Hence

$\mathbb{R}^3$ & W are isomorphic.

$\mathbb{R}^4$, $P_3(\mathbb{R})$, $M_{2\times 2}(\mathbb{R})$ are isomorphic.

4.

(a) The change of coordinate matrix from $\mathcal{B}$ to $\mathcal{C}$ is the matrix

$$Q = \left[I_{P_1(\mathbb{R})} \right]_{\mathcal{B}}^{\mathcal{C}}$$

$$= \left[[1+x]_{\mathcal{C}} \,\middle|\, [1-x]_{\mathcal{C}} \right]$$

$$= \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

(b) The change of coordinate matrix for $\mathcal{C}$ to $\mathcal{B}$ is

$$Q^{-1} = \frac{1}{\det Q} \begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{1}{(-2)} \begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}.$$

(c) $T(1+x) = (3 + 3x) = 3(1+x) + 0(1-x).$

$T(1-x) = -1 + 1x = 0(1+x) + (-1)(1-x).$

Hence

$$[T]_{\mathcal{B}} = \left[[T(1+x)]_{\mathcal{B}} \quad [T(1-x)]_{\mathcal{B}} \right]$$

$$= \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}$$

(d) $T(1) = 1 + 2x$,
$T(x) = 2 + x$, &

$$[T]_{\mathcal{C}} = \begin{bmatrix} [T(1)]_{\mathcal{C}} & [T(x)]_{\mathcal{C}} \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}.$$

(e) Since $Q = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ is the change of basis matrix from $\mathcal{B}$ to $\mathcal{C}$ we have

$$[T]_{\mathcal{B}} = Q^{-1}[T]_{\mathcal{C}} Q$$

by Theorem 2.23.

Check (OPTIONAL)

$$Q^{-1}[T]_{\mathcal{C}} Q = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$
$$= \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix} \begin{bmatrix} 3 & -1 \\ 3 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix} = [T]_{\mathcal{B}}$$

which confirms ~~are~~ our result.