MAS 4105 — Quiz 6 — Fall 2014
Monday, November 17
NAME:

Instructions: all work should be written in a proper and coherent manner, and in a way that any student in the class can follow your work. Show all necessary working and reasoning. When giving proofs your reasoning should be clear.

TOTAL 20 points + 2 bonus points.

1. [1+1+1+1+1 = 5 pts]

(a) Complete Definition: An elementary matrix is a __square__ matrix which can be obtained by performing __a single__ elementary __row or column__ operation on __I__.

(b) Complete Definition: Let $A \in M_{m\times n}(F)$. The rank of A (denoted by rank(A)) is defined to be the __rank of the linear transformation $L_A : F^n \to F^m$ by $L_A(\vec{x}) = A\vec{x}$__.

(c) Complete Theorem: Let $A \in M_{n\times n}(F)$. Then A is invertible if and only if rank(A) = __n__.

(d) Complete Proposition: Let $A \in M_{n\times n}(F)$. If A is invertible then A^t is __invertible__, and

$$(A^t)^{-1} = \underline{(A^{-1})^t}.$$

(e) Complete Theorem: Let $A \in M_{m \times n}(F)$ and suppose $\text{rank}(A) = r$. Then

(i) $r \le$ m and $r \le$ n.

(ii) A can be transformed into a matrix of the form

$$D = \left(\begin{array}{c|c} I_r & O \\ \hline O & O \end{array}\right)$$

by elementary row & column operations

(iii) Further, there are invertible matrices P, Q such that

$D =$ PAQ.

2. [2+3=5 pts]

Let $B = \begin{pmatrix} 1 & 0 & 3 \\ 1 & -2 & 1 \\ 1 & -3 & 1 \end{pmatrix}$, $C = \begin{pmatrix} 1 & 0 & 3 \\ 0 & -2 & -2 \\ 1 & -3 & 1 \end{pmatrix}$.

(i) Find an elementary operation that transforms B into C.

(ii) Find a 3×3 matrix E such that $C = EB$.

$B \xrightarrow{-R_1 + R_2} C$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = E$$

and $C = EB$.

3. [5 pts]

Let A be an $m \times n$ matrix and suppose Q is an $n \times n$ invertible matrix.

Prove that $\text{rank}(AQ) = \text{rank}(A)$.

[Hint: First prove that $R(L_{AQ}) = R(L_A)$.]

Let $\vec{y} \in R(L_{AQ})$. Then $\vec{y} = L_{AQ}(\vec{x}) = AQ(\vec{x})$ for some $\vec{x} \in F^n$.

Then $\vec{y} = A\vec{z}$ where $\vec{z} = Q\vec{x}$.

So $\vec{y} \in R(L_A)$.

Therefore $R(L_{AQ}) \subset R(L_A)$.

Now suppose $\vec{y} \in R(L_A)$.

Then

$$\vec{y} = A\vec{x} \text{ for some } \vec{x} \in F^n$$

This time let

$$\vec{z} = Q^{-1}\vec{x} \qquad \text{(since } Q \text{ is invertible)}$$

Then $\vec{z} \in F^n$ &

$$\begin{aligned} L_{AQ}(\vec{z}) = AQ(\vec{z}) &= AQQ^{-1}\vec{x} \\ &= AI\vec{x} \\ &= A\vec{x} \\ &= L_A(\vec{x}) = \vec{y}. \end{aligned}$$

So $\vec{y} = L_{AQ}(\vec{z})$ & $\vec{y} \in R(L_{AQ})$.

Therefore $R(L_A) \subset R(L_{AQ})$.

Hence $R(L_{AQ}) = R(L_A)$. &

$\text{rank}(AQ) = \dim R(L_{AQ}) = \dim R(L_A) = \text{rank}(A)$.

$\square$

(p.4)

4. [2+3=5 pts]

Let

$$P = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix} \in M_{3\times3}(\mathbb{R}).$$

Explain why P is invertible and find P^{-1}!

$$(P|I) = \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 1 & 1 & -2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{-R_1+R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{R_3+R_2,\ R_3+R_1} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

P is invertible since P can be row reduced to $\underline{I}$.

Also

$$P^{-1} = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

Check: $PP^{-1} = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$= \underline{I}$

(p.5)

5. [2 bonus points]

Let A be an $m \times n$ matrix with rank m. Prove that there exists an $n \times m$ matrix such that $AB = I_m$.

$L_A : F^n \to F^m$. Since $\text{rank}(A) = \dim(R(L_A)) = m$

$R(L_A) = F^m$ & L_A is onto.

Let $\varepsilon = \{\vec{e_1}, \vec{e_2}, \ldots, \vec{e_m}\}$ be the standard basis of F^m. Since L_A is onto for each $1 \le j \le m$ there is a $\vec{v_j} \in F^n$ such that

$$L_A(\vec{v_j}) = A\vec{v_j} = \vec{e_j}.$$

Let $B = [\vec{v_1} \mid \vec{v_2} \mid \cdots \mid \vec{v_m}]$ (is a $n \times m$ matrix),

and

$$AB = A[\vec{v_1} \mid \vec{v_2} \mid \cdots \mid \vec{v_m}]$$

$$= [A\vec{v_1} \mid A\vec{v_2} \mid \cdots \mid A\vec{v_m}]$$

$$= [\vec{e_1} \mid \vec{e_2} \mid \cdots \mid \vec{e_m}]$$

$$= I_m,$$

$\square$

3. (PROOF 2)

Let A be $m \times n$, $\text{rank}(A) = m$ & Q $n \times n$ & invertible.
Then

$$L_A : F^n \longrightarrow F^m \qquad L_Q : F^n \longrightarrow F^n$$

$$L_{AQ} : F^n \longrightarrow F^m.$$

Since $\text{rank}(A) = \dim R(L_A) = m = \dim F^m$,

$$R(L_A) = F^m.$$

Since Q is invertible, L_Q is invertible & onto so that

$$R(L_Q) = L_Q(F^n) = F^n.$$

$$R(L_{AQ}) = L_{AQ}(F^n) = L_A(L_Q(F^n))$$

$$= L_A(F^n)$$

$$= R(L_A).$$

Hence

$$\text{rank}(AQ) = \dim R(L_{AQ}) = \dim R(L_A) = \text{rank}(A).$$