MAS 4105 – GROUP PRACTICE EXAM (Quiz 7)

Wednesday, Dec 3, 2014

Names: 1.

2.

3.

4.

5.

~~Solution~~

Instructions: all work should be written in a proper & coherent manner, and in a way any student in the class can follow your work. Show all necessary working and reasoning. When giving proofs your reasoning should be clear.

TOTAL: 84 pts (To be rescaled to 20 pts)

1. [1+1+1+1+2+1+1+3+1+2 = 14 pts]

(a) Complete Definition: Let $A \in M_{m\times n}(F)$. The rank of A (denoted by rank(A)) is defined to be the rank of the linear transformation $L_A : F^n \to F^m$ by $L_A(\vec{x}) = A\vec{x}$.

(b) Complete Definition: An elementary matrix is a square matrix which can be obtained by performing a single elementary row or column operation on I.

(c) Complete Proposition: Let $A \in M_{n\times n}(F)$. Then A is invertible if and only if rank$(A) = n$.

(d) Complete: <u>Proposition</u>: Let $A \in M_{n\times n}(F)$. If A is invertible then A^t is invertible, and

$$(A^t)^{-1} = (A^{-1})^t.$$

(e) Complete <u>Theorem</u>: Let $A \in M_{m\times n}(F)$ and suppose rank $(A) = r$. Then

(i) $r \leqslant m$ and $r \leqslant n$.

(ii) A can be transformed into a matrix of the form

$$D = \left(\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array}\right)$$

by elementary rows & column operations.

(iii) Further, there are invertible matrices P, Q such that

$$D = PAQ.$$

(f) Complete <u>Proposition</u>: Elementary row operations do not change the linear relationships between columns

(g) Complete <u>Definition</u> (Permutation Defn of Determinant)
Let $A = (a_{ij})$ be an $n \times n$ matrix.
The determinant of A (denoted by $\det(A)$) is

$$\det(A) = \sum_{\sigma \in S_n} \operatorname{sign}(\sigma)\, a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)}.$$

where S_n is set of permutation $\sigma : [n] \to [n]$.

(h) Complete Definition of Determinant Function:

A determinant function is a function $\det : M_{n\times n}(F) \to F$ defined for each n satisfying the following:

(i) If A has two identical rows then

$$\det(A) = \underline{0}.$$

(ii) If A^* is obtained from A by multiplying a single row of A by a scalar α, then

$$\det(A^*) = \underline{\alpha \det(A)}.$$

(iii) If A, A^*, A^{**} are identical except possibly in their i-th row and if ith row of A^{**} = ith row of A + ith row of A^* then

$$\det(A^{**}) = \underline{\det(A) + \det(A^*)}.$$

Alternatively, write

$$\det(A) = \det(\vec{\alpha}_1, \vec{\alpha}_2, \ldots, \vec{\alpha}_n)$$

where the $\vec{\alpha}_i$ are the $\underline{\text{rows}}$ of A. Then

$$\det(\vec{\alpha}_1, \vec{\alpha}_2, \ldots, \vec{\beta}_i + \vec{\gamma}_i, \vec{\alpha}_{i+1}, \ldots, \vec{\alpha}_n)$$
$$= \underline{\det(\vec{\alpha}_1, \vec{\alpha}_2, \ldots, \vec{\beta}_i, \alpha_{i+1}, \ldots, \vec{\alpha}_n) + \det(\vec{\alpha}_1, \vec{\alpha}_2, \ldots, \vec{\gamma}_i, \alpha_{i+1}, \ldots, \vec{\alpha}_n)}$$

(iv) $\det(I_n) = \underline{1}$.

(i) Complete Definition. Let V be a finite dimensional vector space over a field F where $1 \le \dim V = n$. A linear operator $T : V \to V$ is diagonalizable if V has $\underline{\text{an ordered basis } \mathcal{B}}$ such that

$$[T]_{\mathcal{B}} = [T]_{\mathcal{B}}^{\mathcal{B}} = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_n \end{bmatrix}$$

is a $\underline{\text{diagonal}}$ matrix.

(j) Complete Definitions: Let $T: V \to V$ be linear.
A vector $\vec{v}$ in V is an eigenvector of T
if $\vec{v} \neq \vec{0}$ and there is a scalar $\lambda \in F$
such that
$$T(\vec{v}) = \lambda \vec{v}.$$
The scalar $\lambda \in F$ is called an eigenvalue of T if
there is a vector $\vec{v}$ in V, $\vec{v} \neq \vec{0}$ such that
$$T(\vec{v}) = \lambda \vec{v}.$$

2. $[4 \times 5 = 20$ pts$]$

(a) Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 4 & 5 \\ -1 & -1 & -1 \end{bmatrix}$

Explain why A is not invertible.

(b) Let $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ -1 & -1 & 2 \end{bmatrix}$

Explain why B is invertible and find B^{-1}.

(c) Let the reduced row echelon form of A be
$\begin{bmatrix} 1 & -3 & 0 & 4 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Let $\vec{a}_1 = \begin{bmatrix} 1 \\ -2 \\ -1 \\ 3 \end{bmatrix}$, $\vec{a}_3 = \begin{bmatrix} -1 \\ 1 \\ 2 \\ -4 \end{bmatrix}$.

Find A if the first and third columns of A are $\vec{a}_1$, $\vec{a}_3$ (respectively).

(d) Let

$$\vec{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ -1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 2 \\ -2 \\ 0 \\ -2 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \quad \vec{v}_4 = \begin{bmatrix} 2 \\ 1 \\ -3 \\ -2 \end{bmatrix}, \quad \vec{v}_5 = \begin{bmatrix} 1 \\ -2 \\ 2 \\ 0 \end{bmatrix}$$

be vectors in $\mathbb{R}^4$.

Let

$$W = \operatorname{span}(\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4, \vec{v}_5\}).$$

Let

$$A = [\, \vec{v}_1 \mid \vec{v}_2 \mid \vec{v}_3 \mid \vec{v}_4 \mid \vec{v}_5 \,].$$

You are given that

$$R = \begin{bmatrix} 1 & 2 & 0 & 2 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

is the reduced row echelon form of A.

Find a basis for W.

3. Do Four parts [$4 \times 10 = 40$ pts]

(a) Let A be an $m \times n$ matrix and B be a $n \times p$ matrix.

Prove that

$$N(L_B) \subset N(L_{AB})$$

Hence using the Dimension Theorem (or otherwise) prove that

$$\operatorname{rank}(AB) \leq \operatorname{rank}(B).$$

(b) Suppose det is a determinant function & $A \in M_{n \times n}(F)$
Suppose A^* is obtained from A by
the Type III Elementary row operation $\alpha R_i + R_j$.
Prove using definition (See Qu 1 (h)) that
$$\det(A^*) = \det(A).$$

(c) Let $A \in M_{n \times n}(F)$. Let $\lambda \in F$.
Prove that λ is an eigenvalue of A
if and only if $\det(A - \lambda I) = 0$.

(d) Let $A, B \in M_{n \times n}(F)$.
Prove that if A is similar to B then
A and B have the same characteristic polynomial.

(e) Let $A \in M_{n \times n}(F)$. Prove that if A is invertible
then A^t is invertible and
$$(A^t)^{-1} = (A^{-1})^t.$$

4. [3+7=10 pts]
Let $A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & -1 \\ -1 & 0 & 3 \end{bmatrix}$

(i) Show that $\det(A - \lambda I) = (2-\lambda)^2(3-\lambda)$.
(ii) Show that A is diagonalizable and
find a matrix P such that $P^{-1}AP$ is diagonal.

Soln.

2.

(a) $A = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 4 & 5 \\ 7 & 7 & -1 \end{bmatrix} \xrightarrow[R_1+R_3]{-3R_1+R_2} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

So

$\operatorname{rank}(A) = 2 < 3$ & A is not invertible (by Thm. Qu. 1(c)).

(b) $[B \mid I] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ -1 & -1 & 2 & 0 & 0 & 1 \end{array}\right]$

$\xrightarrow{R_1+R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & -1 & 2 & 1 & 0 & 1 \end{array}\right]$

$\xrightarrow{R_2+R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{array}\right]$

$\xrightarrow{R_3+R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{array}\right]$

B is invertible since B is row equivalent to I, and ~~hence~~ we see

$$B^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

(c) Let $\vec{r_1}, \vec{r_2}, \vec{r_3}, \vec{r_4}$ be the columns of the echelon form matrix R of A.

We see

$$\vec{r_2} = -3\vec{r_1} \quad \& \quad \vec{r_4} = 4\vec{r_1} + 3\vec{r_3}.$$

So by Prop. (Qu. 1 (f)),

$$\vec{a_2} = -3\vec{a_1} = -3\begin{pmatrix}1\\-2\\-1\\3\end{pmatrix} = \begin{pmatrix}-3\\6\\3\\-9\end{pmatrix},$$

$$\vec{a_4} = 4\vec{a_1} + 3\vec{a_3} = 4\begin{pmatrix}+1\\-2\\-1\\3\end{pmatrix} + 3\begin{pmatrix}-1\\1\\2\\-4\end{pmatrix}$$

$$= \begin{pmatrix}1\\-5\\2\\0\end{pmatrix} \quad \&$$

$$A = \begin{bmatrix}1 & -3 & -1 & 1\\-2 & 6 & 1 & -5\\-1 & 3 & 2 & 2\\3 & -9 & -4 & 0\end{bmatrix}.$$

(d) Observe that $W = \text{ColumnSpace}(A)$.

~~A basis~~ The columns of A corresponding to columns in R with leading entries will be a basis for the column space.

Hence

$\mathcal{B} = \{\vec{v_1}, \vec{v_3}, \vec{v_5}\}$ is a basis for W

3.

(a) Let $A \in M_{m\times n}(F)$, $B \in M_{n\times p}(F)$ so that

$$AB \in M_{m\times p}(F). \quad \text{Then}$$

$$L_B : F^p \to F^n, \quad L_{AB} : F^p \to F^m.$$

Let $\vec{x} \in N(L_B)$. Then $\vec{x} \in F^p$ & $L_B(\vec{x}) = B\vec{x} = \vec{0}$.

So

$$L_{AB}(\vec{x}) = AB(\vec{x}) = A(B\vec{x}) = A(\vec{0}) = \vec{0},$$

&

$$\vec{x} \in N(L_{AB}). \text{ Therefore}$$

$$N(L_B) \subset N(L_{AB}), \text{ and}$$

$N(L_B)$ is a subspace of $N(L_{AB})$, and

$$\dim N(L_B) \leq \dim N(L_{AB}). \quad (*).$$

By The Dimension Thm,

$$\dim(N(L_B)) + \dim R(L_B) = p,$$

$$\dim(N(L_{AB})) + \dim R(L_{AB}) = p.$$

So $\text{rank}(B) - \text{rank}(AB) = \dim R(L_B) - \dim R(L_{AB})$

$$= (p - \dim N(L_B)) - (p - \dim N(L_{AB}))$$

$$= \dim N(L_{AB}) - \dim N(L_B) \geq 0, \text{ (by } (*))$$

Therefore,

$$\text{rank}(AB) \leq \text{rank}(B).$$

(b) Let $\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_n$ denote the rows of A.
Then
$A^* = (\vec{r}_1, \ldots, \vec{r}_i, \ldots, \alpha\vec{r}_i + \vec{r}_j, \ldots, \vec{r}_n)$.

$\det(A^*) = \det(\vec{r}_1, \ldots, \vec{r}_i, \ldots, \alpha\vec{r}_i, \ldots, \vec{r}_n)$
$\quad + \det(\vec{r}_1, \ldots, \vec{r}_i, \ldots, \vec{r}_j, \ldots, \vec{r}_n)$ (by Defn. Qu 1(h)(iii))

$= \alpha \det(\vec{r}_1, \ldots, \vec{r}_i, \ldots, \vec{r}_i, \ldots, \vec{r}_n)$ (by Defn Q 1(h) (ii))
$\quad + \det(A)$

$= \alpha \cdot 0 + \det(A)$ (by Defn Qu 1(h)(i))
$= \det(A)$. So
$$\det(A^*) = \det(A).$$

(c) Let $A \in M_{n\times n}(F)$ & $\lambda \in F$.

λ is an eigenvalue of $A \iff A\vec{x} = \lambda\vec{x}$ for some $\vec{x} \in F^n$, $\vec{x} \neq \vec{0}$.
$\iff (A - \lambda I)\vec{x} = \vec{0}$ for some $\vec{x} \in F^n$, $\vec{x} \neq \vec{0}$.
$\iff N(T) \neq \{\vec{0}\}$ where $T: F^n \to F^n$ by $T(\vec{x}) = (A - \lambda I)\vec{x}$.
$\iff \operatorname{rank}(T) \neq \dim F^n = n$ (since $F^n = F^n$).
$\iff T$ is not invertible
$\iff [T]_\varepsilon = A - \lambda I$ (where ε is the standard basis of F^n)
is not invertible
$\iff \det(A - \lambda I) = 0$.

(d) Suppose A is similar to B. Then

$$A = Q^{-1} B Q$$

for some invertible matrix Q.
Characteristic Polynomial of A

$$= \det(A - tI)$$

$$= \det(Q^{-1}BQ - tI)$$

$$= \det\left(Q^{-1}(BQ - Q(tI))\right)$$

$$= \det(Q^{-1}) \det(BQ - t(QI))$$

$$= \det(Q^{-1}) \det(\cancel{BQ} (B - tI)Q)$$

$$= \det(Q^{-1}) \det(B - tI) \det(Q).$$

$$= \det(B - tI)$$

(since Q is invertible and $\det(Q^{-1}) = \frac{1}{\det(Q)}$.)

$$= \text{Characteristic Polynomial of } B.$$

(e) Let $A \in M_{n \times n}(F)$. Suppose A is invertible.
Then

$$A A^{-1} = A^{-1} A = I.$$

So $(AA^{-1})^t = (A^{-1}A)^t = I^t = I.$

$$(A^{-1})^t A^t = A^t (A^{-1})^t = I.$$

Therefore

$$\cancel{(A^t)^t A^t}$$

$$A^t (A^{-1})^t = (A^{-1})^t A^t = I.$$

Hence A^t is invertible & by uniqueness,

$$(A^t)^{-1} = (A^{-1})^t.$$

4.

(i)

$$\det(A-\lambda I)=\det\begin{pmatrix}2-\lambda & 0 & 0\\ 1 & 2-\lambda & -1\\ -1 & 0 & 3-\lambda\end{pmatrix} \qquad \begin{matrix}+ & - & +\\ - & + & -\\ + & - & +\end{matrix}$$

$$=\ (2-\lambda)\det\begin{pmatrix}2-\lambda & -1\\ 0 & 3-\lambda\end{pmatrix} - 0\det(\ ) + 0\det(\ )$$

(by a Cofactor Expansion along 1^{st} row).

$$=(2-\lambda)\left((2-\lambda)(3-\lambda)-0\right)$$

$$=(2-\lambda)^2(3-\lambda).$$

(ii) $\det(A-\lambda I)=(2-\lambda)^2(3-\lambda)=0$ for $\lambda=2,3$.

A has two eigenvalues $\lambda=2,3$.

To show A is diagonalizable we must show that A has 3 linearly independent eigenvectors.

$\boxed{\lambda=2}$ We solve $(A-2I)\vec{x}=\vec{0}$

$$\Leftrightarrow \left[\begin{array}{ccc|c}0 & 0 & 0 & 0\\ 1 & 0 & -1 & 0\\ -1 & 0 & 1 & 0\end{array}\right]$$

$$\Leftrightarrow \left[\begin{array}{ccc|c}1 & 0 & -1 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{array}\right]$$

x_2, x_3 are free & $x_1=x_3$

$$\vec{x}=\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}x_3\\x_2\\x_3\end{pmatrix}=x_2\begin{pmatrix}0\\1\\0\end{pmatrix}+x_3\begin{pmatrix}1\\0\\1\end{pmatrix}.$$

$\vec{v}_1=\begin{pmatrix}0\\1\\0\end{pmatrix}$, $\vec{v}_2=\begin{pmatrix}1\\0\\1\end{pmatrix}$ are lin. indep. eigenvects with $\lambda=2$

$\boxed{\lambda=3}$ We solve $(A-3I)\vec{x}=\vec{0}$

$$\Leftrightarrow \left[\begin{array}{ccc|c} -1 & 0 & 0 & 0 \\ 1 & -1 & -1 & 0 \\ -1 & 0 & 0 & 0 \end{array}\right]$$

$$\Leftrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\Leftrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

x_3 is free, $x_1=0$, $x_2=-x_3$;

$$\vec{x}=\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}=\begin{pmatrix} 0 \\ -x_3 \\ x_3 \end{pmatrix}=x_3\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix},$$

So $\vec{v}_3=\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$ is an eigenvector with eigenvalue $\lambda=3$

We show $\vec{v}_1,\vec{v}_2,\vec{v}_3$ are linearly indept. Suppose

$$x_1\vec{v}_1+x_2\vec{v}_2+x_3\vec{v}_3=\vec{0}$$

$$\Leftrightarrow \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \Leftrightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$\Leftrightarrow$ $x_1=x_2=x_3=0$.

So $\mathcal{B}=\{\vec{v}_1,\vec{v}_2,\vec{v}_3\}$ is linearly indept. & hence a basis of $\mathbb{R}^3$ of eigenvectors. Hence A is diagonalizable &

$$P^{-1}AP=\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \text{ where } P=\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}=[\vec{v}_1|\vec{v}_2|\vec{v}_3].$$