* What you should know for QUIZ 4

Review NOTES & PROBLEMS

2.2 The Matrix Representation of a Linear Transformation

Definition Let V be a finite dimensional vector space over a field F. An ordered basis is ____

$\mathcal{B}$ = ____________ (ordered)

where the elements of the ____ form a ____ for V.

NOTE: In an ordered basis, ______ matters.

Definition

Let $\mathcal{B} = [\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n]$ be an ordered basis of a vector space V over a field F. Let $\vec{v} \in V$. Then there are ________ scalars $c_1, c_2, \ldots, c_n \in$ ____ such that

$\vec{v} =$

The vector $\begin{bmatrix} \\ \\ \\ \end{bmatrix} \in$ ____ is called

the coordinate vector of $\vec{v}$ relative to the basis $\mathcal{B}$ and we write

$$[\_\_]_{\_\_\_} = \begin{bmatrix} \\ \\ \\ \\ \end{bmatrix}.$$

Example Let $\mathcal{B} = [\vec{b}_1, \vec{b}_2]$

where $\vec{b}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\vec{b}_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

(i) Explain why $\mathcal{B}$ is a ordered basis of $\mathbb{R}^2$.

(ii) Find the coordinate vector of $\vec{v} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ relative to the basis $\mathcal{B}$.

(iii) Find the coordinate vector of $\vec{v} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ relative to the standard basis $\mathcal{E} = [\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}]$ of $\mathbb{R}^2$.

Example You are given that

$\mathcal{B} = [\, p_1(x) = 1 - 2x - 2x^2,\ p_2(x) = 1 + x + x^2,$
$\quad p_3(x) = 2 - 4x + 5x^2 \,]$

is an ordered basis for $\mathbb{P}_2(\mathbb{R})$.

Let $p(x) = 5 - 12x + 11x^2$.

(i) Find the coordinate vector $[p(x)]_{\mathcal{B}}$.

(ii) What is the coordinate vector of $p(x)$ relative to $\mathcal{E} = [1, x, x^2]$?

Definition: Suppose $\mathcal{B} = [\vec{v_1}, \vec{v_2}, \ldots, \vec{v_n}]$ is an ordered basis of V & $\mathcal{C} = [\vec{w_1}, \vec{w_2}, \ldots, \vec{w_m}]$ is a ordered basis of W & $T: V \to W$ is linear. Then the matrix of the transformation T with respect to (or relative to) the bases $\mathcal{B}$ and $\mathcal{C}$ is denoted & defined by

$$\left[\quad\quad\right]_{---}^{---} = \left[\ \left[\quad\right]_{---} \cdot \ \middle|\ \left[\quad\right]_{---} \ \middle|\ \cdots \ \middle|\ \left[\quad\right]\ \right]$$

Note: $[T]_{\mathcal{B}}^{\mathcal{C}}$ is a $\_\_ \times \_\_$ matrix A that satisfies

$$T(\_\_) = \_\_\_\_\_\_$$

for $\_\_\_\_\_\_\_\_$. Observe that

$$\left[T(\vec{v_j})\right]_{---} = \left[\quad\right]$$

for $\_\_\_\_$ where $A = [a_{ij}]_{----}^{----} = \left[\quad\right]_{---}^{---}$.

Example Let $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ by
$T(a_1, a_2) = (a_1 + a_2, a_1 - a_2, a_1)$.

Let $\mathcal{B} = \left[\vec{b}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \vec{b}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right]$ be the standard basis of $\mathbb{R}^2$, &

$$\mathcal{C} = \left[\vec{c}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \vec{c}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \vec{c}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right]$$

be the standard basis of $\mathbb{R}^3$. Find $[T]_{\mathcal{B}}^{\mathcal{C}}$
assuming T is linear.

Example Let $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ by $T(a_1, a_2) = (4a_1 + 2a_2, -a_1 + a_2)$.
Let $\mathcal{B} = \left[\vec{b}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \vec{b}_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \right]$. Then $\mathcal{B}$ is a basis for $\mathbb{R}^2$.

Find $[T]_{\mathcal{B}}^{\mathcal{B}}$ assuming T is linear.

Example Let $T : P_2(\mathbb{R}) \longrightarrow P_1(\mathbb{R})$ by

$T(p(x)) = p'(x)$. Then T is linear.

Let $\mathcal{B} = [1, x, x^2]$ be the standard basis of $P_2(\mathbb{R})$, & $\mathcal{C} = [1, x]$ be standard basis of $P_1(\mathbb{R})$.

Find $[T]_{\mathcal{B}}^{\mathcal{C}}$.

(p. 7)

Definition Let V, W be vector spaces over a field F.

Let $\mathcal{L}(V, W)$ denote the set of ______ transformations T: ___ $\longrightarrow$ ___. Then $\mathcal{L}(V, W)$ is a vector space over F with addition & scalar multiplication defined as follows:

Let $T, U \in$ _____ and $c \in$ ___, so that T: ___ $\rightarrow$ ____ & U: ___ $\rightarrow$ ___ are ______ transformations.

Then $T+U$: ___ $\longrightarrow$ ___ is defined by

$(T+U)($__$) =$ ________

and

cT : ___ $\longrightarrow$ ___ is defined by

$(cT)($___$) =$ _____,

for $\vec{v} \in$ ____.

Note $\mathcal{L}(V, W) \subset$ _____.

① The zero transformation is $T_0 : V \rightarrow W$ by $T_0(\vec{v}) =$ ____ for $\vec{v} \in V$.

Let $\vec{u}, \vec{v} \in V$ & $c \in F$.

Then $T_0(\vec{u} + \vec{v}) =$

$=$

$=$

$T_0(c\,\vec{u}) =$

$=$

$=$

Hence T_0 is linear & $T_0 \in$ ______.

(2) Let $T, U \in \mathcal{L}(V, W)$.

We show $T+U \in \mathcal{L}(V, W)$.

Let $\vec{u}, \vec{v} \in V$ & $c \in F$. Then

$(T+U)(\vec{u}+\vec{v}) =$

$=$

$=$

$=$

$(T+U)(c\vec{u}) =$

$=$

$=$

$=$

Hence $T+U$ is _____ & $T+U \in$ _____

(3) Let $d \in F$.

We show $dT \in$ _____.

Let $\vec{u}, \vec{v} \in V$ & $c \in F$. Then

$(dT)(\vec{u}+\vec{v}) =$

$=$

$=$

$=$

$(dT)(c\vec{u}) =$

$=$

$=$

$=$

Hence dT is _____ & $dT \in$ _____.

(p. 9)

Therefore $\mathcal{L}(V, W)$ is a subspace of ________,
and is hence a vector space over F.

Example Let $T: P(\mathbb{R}) \to P(\mathbb{R})$ by $T(p(x)) = p'(x)$
and $U: P(\mathbb{R}) \to P(\mathbb{R})$ by $U(p(x)) = x\,p(x)$.
Then T & U are ________ transformations.
Hence
$$(T+U): \_\_\_ \longrightarrow \_\_\_\_ \text{ is}$$
________ where
$$(T+U)(p(x)) = \_\_\_\_\_\_\_\_.$$

Theorem 2.8
Let V, W be finite dimensional vector spaces (over F) with ordered bases $\mathcal{B}$ and $\mathcal{C}$ respectively. Let $T, U \in \mathcal{L}(V, W)$. Then

(a) $$[T+U]_{\mathcal{B}}^{\mathcal{C}} = \_\_\_\_\_\_\_\_\_\_\_,$$ and

(b) $$[\alpha T]_{\mathcal{B}}^{\mathcal{C}} = \_\_\_\_\_\_,$$

for all $\alpha \in$ ____.

Proof

(a) Let $\mathcal{B} = [\vec{v_1}, \vec{v_2}, \ldots, \vec{v_n}]$ be a ordered basis of V,

$\mathcal{C} = [\vec{w_1}, \vec{w_2}, \ldots, \vec{w_m}]$ be a ordered basis of W.

Let $[T]_{\mathcal{B}}^{\mathcal{C}} = (a_{ij})_{\substack{1 \le i \le \_\_\_ \\ 1 \le j \le \_\_\_}}$,

$[U]_{\mathcal{B}}^{\mathcal{C}} = (b_{ij})_{\substack{1 \le i \le \_\_\_ \\ 1 \le j \le \_\_\_}}$

Then for $1 \le j \le n$,

$T(\vec{v_j}) =$

$U(\vec{v_j}) =$

$(T + U)(\vec{v_j}) =$

$=$

$=$

Thus $[T + U]_{\mathcal{B}}^{\mathcal{C}} =$

$=$

=

Excersise Prove (b).

PAST EXAM PROBLEMS

(1) Let $T: M_{2\times 2}(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ by $T(A) = A^t$.
Then T is linear.
Let

$$\mathcal{B} = \left[B_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, B_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, B_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, B_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right]$$

be the standard basis of $M_{2\times 2}(\mathbb{R})$

Find $[T]_{\mathcal{B}} = [T]_{\mathcal{B}}^{\mathcal{B}}$

(2) Let $T: P_2(\mathbb{R}) \to \mathbb{R}$ by $T(p(x)) = p(2)$.
Then T is linear.

Compute $[T]_{\mathcal{B}}^{\mathcal{C}}$

where $\mathcal{B} = [1, x, x^2]$ & $\mathcal{C} = [1]$.

(p.13)

Suggested Homework of §2.2 # 2, 3, 4, 5, 8, 9, 10.

2.3

(P.1)

2.6 Composition of Linear Transformations & Matrix Multiplication.

Theorem 2.9

Let V, W, Z be vector spaces over F.
Let

$T: V \longrightarrow$ ___ and U: ___ $\longrightarrow$ ___

be linear. Then UT: ___ $\longrightarrow$ ___
defined by $UT($__$) =$ ______
is ________.

PROOF:

Let $\vec{v_1}, \vec{v_2} \in V$ and $c \in F$. Then

$(U\ T)(\vec{v_1} + \vec{v_2}) =$
$=$
$=$
$=$

$(UT)(c\vec{v_1}) =$
$=$
$=$
$=$

Hence UT: ___ $\longrightarrow$ ___ is ______. □

Definition

Let A be a $m \times$ __ matrix and
B be a __ $\times p$ matrix. Then the
product AB is a __ $\times$ __ matrix defined by

$(AB)_{ij} =$ ______

For $1 \le i \le$ ___ and $1 \le j \le$ ___.

Example

Let $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, $B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$.

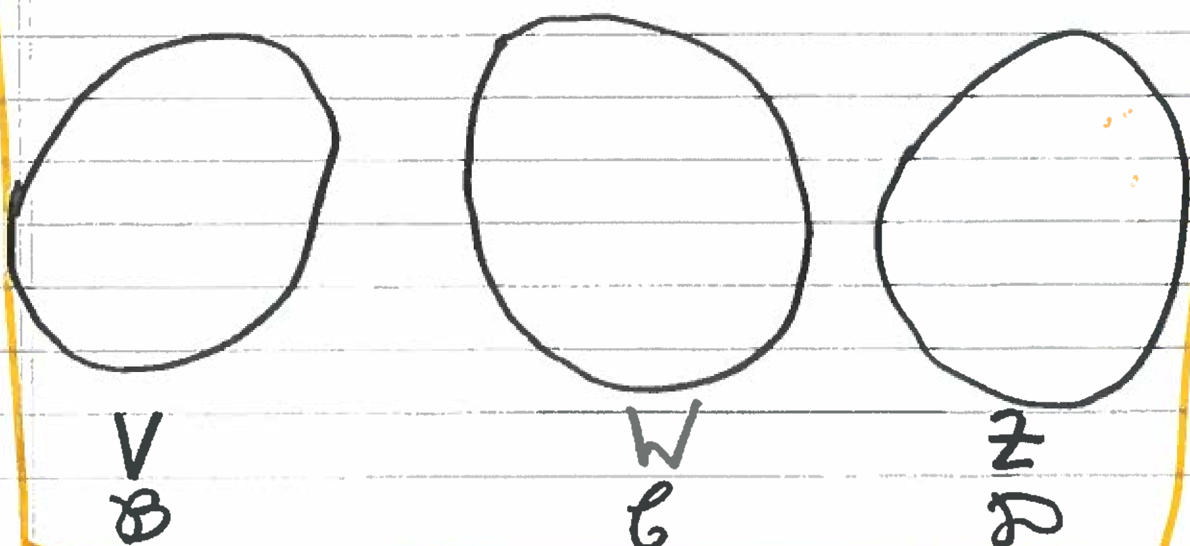
Then

$$AB = \begin{pmatrix} \quad & \quad \end{pmatrix}.$$

Theorem 2.11

Let V, W, Z be finite dimensional vector spaces over F with ordered bases $\mathcal{B}, \mathcal{C}, \mathcal{D}$ respectively. Suppose $T: V \longrightarrow$ ___ and $U:$ ___ $\longrightarrow Z$ are linear. Then $UT:$ ___ $\longrightarrow$ ___ is ______ and

$$[UT]^{\cdots}_{\cdots} = [\quad]^{\cdots}_{\cdots} \, [\quad]^{\cdots}_{\cdots}.$$



PROOF. Let

$\mathcal{B} = [\vec{b}_1, \vec{b}_2, \ldots \vec{b}_n]$ be an ordered basis for V,

$\mathcal{C} = [\vec{c}_1, \vec{c}_2, \ldots, \vec{c}_m]$ .. .. W.

$\mathcal{D} = [\vec{d}_1, \vec{d}_2, \ldots, \vec{d}_p]$ .. .. Z.

Let $B = [T]_{\mathcal{B}}^{\mathcal{C}}$, $A = [U]_{\mathcal{C}}^{\mathcal{D}}$.

Then

B is __ x __ and A is __ x __.

$$T(\vec{b}_j) = \sum_{k=1}^{\ldots} B_{kj} \vec{c}_k$$

$$U(\vec{c}_k) = \sum_{i=1}^{\ldots} \ldots\ldots \vec{d}_i$$

$$UT(\vec{b}_j) =$$
$$=$$
$$=$$
$$=$$
$$=$$
$$=$$
$$=$$
$$=$$

Hence $[UT]_{\ldots}^{\ldots} =$

(p. 4)

Theorem 2.13

Let $\vec{v_1}, \vec{v_2}, \ldots, \vec{v_p} \in F^n$ be columns of a matrix B

ie

$B = [\vec{v_1} | \vec{v_2} | \cdots | \vec{v_p}]$ and B is a __ x __ matrix.

Let A be a $m \times$ __ matrix. Then

$$AB = A[\vec{v_1} | \vec{v_2} | \cdots | \vec{v_p}]$$

$$= [\quad | \quad | \cdots | \quad]; \text{ i.e.}$$

(a) Column j of AB = _____.

(b) $\vec{v_j}$ = _____

where ________________.

PROOF: Let $1 \leq j \leq p$.

$\vec{v_j}$ is a __ x __ matrix and

$[\vec{v_j}]_{k1} = B_{kj}$ f $1 \leq k \leq n$.

$A\vec{v_j}$ is a __ x __ matrix.

Entry in Row i of $A\vec{v_j}$

$=$

$\ldots\ldots\ldots\ldots$

$=$ Entry in $\ldots\ldots\ldots\ldots$ of AB.

for $1 \leq i \leq \ldots\ldots$. Hence

Column j of $AB =$ $\_\_\_\_$.

(b) Let $\vec{e_1}, \vec{e_2}, \ldots, \vec{e_p}$ be the standard basis vectors of F^p.

Let $I_p = [\vec{e_1} | \vec{e_2} | \ldots | \vec{e_p}] = (\delta_{ij})$

where $\delta_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{otherwise} \end{cases}$

$1 \leq i, j \leq p$.

$$(BI_p)_{ij} = \sum_{k=1}^{p} B_{ik}\,\delta_{kj} = \_\_\_\_$$

for $1 \leq i \leq \_\_\_$ & $1 \leq j \leq \ldots\ldots$

Hence

$BI_p = \_\_\_\_$

By (a)

$\vec{v}_j$ = Column j of ___

= Column j of ___

= ________. $\square$

Algebraic Properties of Matrices

Let A be an $m \times n$ matrix.
Let B, C be $n \times p$ matrices,
D, E be $q \times m$ matrices.
Then

(1) $A(B+C) =$

(2) $(D+E)A =$

(3) $a(AB) = (aA)B = A(aB)$ for any _____.

(4) $I_m A =$ ___ $= A I_n$

where I_n is the $n \times n$ ____ matrix

$$I_n = \begin{bmatrix} \\ \\ \\ \end{bmatrix} = (\delta_{ij}) \ldots$$

where
$$\delta_{ij} = \begin{cases} \\ \\ \end{cases}$$

(3) Suppose $\dim V = n$ & $\mathcal{B}$ is an ordered basis for V. Then the identity transformation

$I_V : \_\_ \to \_\_$ by $I_V(..) = \ldots$

is _______ and

$$[I_V]_{\mathcal{B}}^{\mathcal{B}} = [I_V]_{\mathcal{B}} = \_\_\_\_ .$$

PROOF (1)

$$[A(B+C)]_{ij} =$$
$$=$$
$$=$$
$$=$$
$$=$$

Hence

(2) Similarly $(D+E)A = \ldots\ldots\ldots$ (EX)

PROOF (4)

$I_m A$ is a __ x __ matrix.

$[I_m A]_{ij} =$

$=$

$=$

for each $1 \leq i \leq \ldots$, & $1 \leq j \leq \ldots$

Hence

$I_m A = \_\_\_$

Similarly,

$A I_n = \_\_\_$ (EX.)

PROOF (5)

Corollary

Let A be an $m \times n$ matrix.
Let $B_1, B_2, \ldots, B_k$ be $\_\_ \times \_\_$ matrices, &
$C_1, C_2, \ldots, C_k$ be $\_\_ \times \_\_$ matrices,
and $a_1, a_2, \ldots\ldots a_n$ be scalars.
Then

(1) $A\left(\sum_{i=1}^{k} a_i B_i\right) =$

(2) $\left(\sum_{i=1}^{k} a_i C_i\right) A =$

Theorem (See Ex. 14(a)).

Let B be an $m \times n$ matrix with columns $\vec{b}_1, \vec{b}_2, \ldots\ldots \in \ldots$.

Let $\vec{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \in F^n$.

Then
$$B\vec{a} = \ldots\ldots\ldots\ldots\ldots\ldots$$

which is a ________________ of the __________ of B.

PROOF: $\vec{a} = a_1\vec{e}_1 + a_2\vec{e}_2 + \cdots + a_n\vec{e}_n$
where $\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n$ are the ______________
____ of _____.

Then

$$\mathcal{B}\,\vec{a} =$$

$$=$$

$$=$$

Since

$\square$

Theorem

Let $\mathcal{B} = [\vec{b_1}, \vec{b_2}, \ldots, \vec{b_n}]$ be an ordered basis of a f.d. vectorspace V. Then

$T: V \to$ ____ by $T(\vec{v}) = [\vec{v}]_{\mathcal{B}}$

is a ________ transformation.

PROOF: Let $\vec{v_1}, \vec{v_2} \in V$ & $c \in F$.
Then let

$$T(\vec{v_1}) =$$

$$T(\vec{v_2}) =$$

So

$$\vec{v_1} =$$

$$\vec{v_2} =$$

$\vec{v_1} + \vec{v_2} =$

$[\vec{v_1} + \vec{v_2}]_{\mathcal{B}} =$

$T(\vec{v_1} + \vec{v_2}) =$

$c\vec{v_1} =$

~~So~~ $[c\vec{v_1}]_{\mathcal{B}} =$

and

$T(c\vec{v_1}) =$

and T is ______. $\square$

<u>Corollary</u> With V, $\mathcal{B}$ as above

$[\alpha_1\vec{v_1} + \alpha_2\vec{v_2} + \dots + \alpha_n\vec{v_n}]_{\mathcal{B}}$

$=$

for

Theorem 2.14

Let V, W be f.d. vector spaces with ordered bases $\mathcal{B}$, $\mathcal{C}$ respectively.
Let $T : V \longrightarrow W$ be linear. Then

$$[\quad\quad]_{\dots} = [T]^{\dots}_{\dots} [\quad]_{\dots}$$

for all $\vec{v} \in V$.

PROOF:

Let $A = [T]^{\mathcal{C}}_{\mathcal{B}} = [\quad\quad\quad]$

Suppose $\vec{v} \in V$ & let

$$[\vec{v}]_{\mathcal{B}} = \vec{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \in \dots$$

Then

$$\vec{v} =$$

$$T(\vec{v}) =$$

$$[T(\vec{v})]_{\mathcal{C}} =$$

$=$

$=$

$=$

$=$

$\square$

Example

Let $\mathcal{B} = \left[\vec{v_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \vec{v_2} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \right]$,

$\mathcal{C} = \left[\vec{w_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \vec{w_2} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right]$.

Let $T : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be linear.
You are given that

$$[T]_{\mathcal{B}}^{\mathcal{C}} = \begin{bmatrix} 3 & 4 \\ -2 & 5 \end{bmatrix}.$$

Find $T(\vec{v})$, where $\vec{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

p.14)

MULTIPLICATION BY A MATRIX

Definition: Let $A \in M_{m \times n}(F)$.
Define
$$L_A : \_\_ \to \_\_ \text{ by } L_A(\vec{x}) = \_\_\_.$$

Theorem Let $A \in M_{m \times n}(F)$.
Then $L_A : \_\_ \to \_\_$ is a _______ transformation.

PROOF: Let $\vec{x} \in F^n$ so that $\vec{x}$ is a $\_\_ \times \_\_$ matrix.

$$L_A(\vec{x}) = \_\_\_\_\_ \text{ is defined \&}$$

is a $\_\_ \times \_\_$ matrix and hence a vector in ___,
so that L_A is well-______.

Let $\vec{x}, \vec{y} \in F^n$ & $c \in F$.

Then $L_A(\vec{x} + \vec{y}) =$
$=$
$=$

also
$L_A(c\vec{x}) =$
$=$
$=$

and L_A is _______. $\square$

(p.16)

Lemma Let $A \in M_{m \times n}(F)$, $B \in M_{n \times p}(F)$ & $\vec{x} \in$ ____.

Then

$$(AB)\vec{x} = \text{\_\_\_\_\_\_\_\_}.$$

PROOF: Let $B = [\vec{b}_1 \mid \vec{b}_2 \mid \cdots \mid \vec{b}_p]$,

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{bmatrix}$$

Then

$$AB =$$

$$(AB)\vec{x} =$$

$$=$$

$$=$$

$$= \qquad \square$$

Theorem 2.15 Let $B, A \in M_{m\times n}(F)$.
Let β be the standard basis of F^n
γ " " " " " F^m.

Then

(1) $[L_A]_\beta^\gamma =$

(2) $L_A = L_B$ iff ______

(3) $L_{(A+B)} =$, $L_{(cA)} =$
for $c \in F$.

(4) If $T: F^n \to F^m$ is linear then there exists a ______ matrix $C \in M_{\dots}(F)$ such that

$T =$, namely $C = \dots$.

(5) If E is a $n \times p$ matrix then $L_{AE} =$

(6) If $m = n$ then $L_{I_n} =$

PROOF:

(1) $L_A(\vec{x}) =$ ____ for $\vec{x} \in$ ___.

So $[L_A]_{\mathcal{B}}^{\mathcal{G}} =$

(where $\mathcal{B} = [\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n]$ is the ________ basis of ___)

$=$

$=$

$=$

$=$

(4) Suppose $T: F^n \to F^m$ is linear.

Then $[T(\vec{x})]_{\ldots} = [\quad]_{\ldots}^{\ldots} [\quad]_{\ldots}$

for $\vec{x} \in$ ___.

(pf)

Let $\left[T(\vec{x})\right]_{\mathcal{C}} =$

and $[\vec{x}]_{\mathcal{B}} =$

Hence

$$T(\vec{x}) =$$

and $T = L_{C}$ where $C =$

Conversely, suppose $T = L_C$ where $C \in M_{m \times n}(F)$. Then by (1),

$$C =$$

Corollary: Let A, B, C be matrices so that $A(BC)$ is defined. Then

$$A(BC) =$$

PROOF.

$$L_A(L_B L_C) =$$

but

$$L_A(L_B L_C) =$$

and

$$(L_A L_B) L_C =$$

Hence

$$(AB)C =$$

by (2) of the Theorem. $\square$

PAST EXAM QUESTIONS

① Let $\mathcal{B} = \left[\begin{pmatrix}1\\2\end{pmatrix}, \begin{pmatrix}1\\3\end{pmatrix}\right]$,

$$\mathcal{C} = [1+x, 1-x].$$

Then
$\mathcal{B}$ is a ordered basis for $\mathbb{R}^2$ &

$\mathcal{C}$ is a ordered basis for $P_1(\mathbb{R})$

Let $\vec{v} \in \mathbb{R}^2$ where $[\vec{v}]_{\mathcal{B}} = \begin{bmatrix}3\\4\end{bmatrix}$.

Suppose $U: \mathbb{R}^2 \to P_1(\mathbb{R})$ is linear and

$$[U]_{\mathcal{B}}^{\mathcal{C}} = \begin{bmatrix}-1 & 1\\-3 & 2\end{bmatrix}.$$

Find $U(\vec{v})$.

(2) Find linear transformation $U, T: F^2 \to F^2$ such that

$$UT = T_0 \quad (\text{the zero transformation})$$

but

$$TU \neq T_0.$$

(p.23)

③ Define $T: P_2(\mathbb{R}) \to \mathbb{R}$ by

$T(p(x)) = p(2)$. Then T is a linear transformation.

(a) Compute $[T]_{\mathcal{B}}^{\mathcal{C}}$ where $\mathcal{B} = [1, x, x^2]$,

$\mathcal{C} = [1]$.

(b) Use your answer to (a) & Theorem 2.14 to compute $T(f(x))$ where $f(x) = 6 - x - 2x^2$.

Check your answer by computing $T(f(x))$ directly.

(p.24).

Suggested Homework (2.3):

1, 2, 3, 4, 9, 11, 12, 13