$\boxed{\text{What you need to know for QUIZ 6}}$

Chapter 3
Elementary Matrix Operations & Systems of Linear Equations

3.1 Elementary matrix operations & elementary matrices

Elementary Row Operations

(1) ________ two rows (______)
(2) Multiply a row by __________________(____)
(3) Add ______________________(______)

Example

$$A = \begin{pmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{pmatrix}$$

$$\xrightarrow{(\quad)} \begin{pmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ & & & & & \end{pmatrix}$$

$$\xrightarrow{(\quad)} \begin{pmatrix} & & & & & \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 1 & & & & & \\ 0 & & & & & \\ 0 & 3 & -6 & 6 & 4 & -5 \end{pmatrix}$$

Elementary Column Operations

(1) ________ two columns (____)
(2) Multiply a column by ____________ (___)
(3) add ______________________ (_____)

Example

$$A = \begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix} \xrightarrow{(\quad)} \begin{pmatrix} 1 & 0 & 7 \\ 2 & & 8 \\ 3 & & 9 \end{pmatrix}$$

Definition

An elementary matrix is a ________ matrix which can be obtaining by performing ____ elementary ______ ____ operation on ____.

Example ①

(1)
$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{-2R_1+R_3} \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix} = E$$

E is a Type __ elementary matrix corresponding to the row operation ______.

NOTE

(1) $I \longrightarrow E$ also corresponds to the column operation ______.

(2) $E \xrightarrow{\qquad} I$

(3) $E^{-1} = \begin{pmatrix} \quad & \quad & \quad \\ & & \\ & & \end{pmatrix}$ is also an elementary matrix & it corresponds to the row operation ___

Exercise: Check that $E^{-1}E = E^{-1}E = I$.

Example (2)

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{(R_1 \leftrightarrow R_2)} E = \begin{pmatrix} \quad & \quad & \quad \\ & & \\ & & \end{pmatrix}$$

E is an elementary matrix of type ___ corresponding to the row operation ___. It also corresponds to the column operation ___

$E^{-1} =$ ___ corresponds to ___

and $E \xrightarrow{\qquad} I$.

Example (3)

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{(3R_2)} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix} = E$$

E is an elementary matrix of type ___ corresponding to the row operation ___. It also corresponds to the column operation ___.

$$E^{-1} = \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix}$$ which corresponds to ____

and $E \xrightarrow{\boxed{}} I$.

<u>Example</u>

Let $E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}$. E is the elementary 3×3 matrix corresponding to the row operation ____.

Let $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$

Then

$$EA = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix} \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

$$= \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix}$$

Observe that

$$A \longrightarrow \_\_\_\_$$

<u>Exercise</u> Show that $A \xrightarrow{(-2C_3 + C_1)} \_\_\_\_$.

<u>Theorem</u> Let $A \in M_{m \times n}(F)$.

(1) Let E be the $m \times m$ elementary matrix corresponding to the elementary row operation $\mathcal{R}$. Then

$$A \xrightarrow{\mathcal{R}} \_\_\_\_.$$

(2) Let E' be the $\_\_ \times \_\_$ elementary matrix corresponding to the elementary column operation $\mathcal{C}$. Then

$$A \xrightarrow{\mathcal{C}} \_\_\_\_.$$

<u>Theorem</u> Elementary matrices are ____ ____, and the inverse is an ____ ____ matrix of the ____ ____.

3.2 The Rank of a Matrix & Matrix Inverse

<u>Definition</u> Let $A \in M_{m \times n}(F)$. The <u>rank</u> of A (denoted by : ________) is defined to be the

____.

In otherwords, $\operatorname{rank}(A) =$ ________________.

<u>Example</u> Let $A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \in M_{2\times 2}(\mathbb{R})$.

Find $\operatorname{rank}(A)$.

$L_A : \mathbb{R}^2 \to \mathbb{R}^2$ by $L_A(\vec{x}) = A\vec{x}$

Theorem 3.3 Let $T: V \to W$ be linear with V, W finite dimensional vector spaces with ordered bases $\mathcal{B}$, $\mathcal{C}$ respectively. Let $A = [T]_{\mathcal{B}}^{\mathcal{C}}$.

Then

$$\text{rank}(T) = \_\_\_\_\_\_\_.$$

PROOF:

Suppose $\{T(\vec{v}_1), T(\vec{v}_2), \ldots, T(\vec{v}_k)\}$ is __________
where each $\vec{v}_j \in$ ____.

We will show that $[T(\vec{v}_1)]_{\mathcal{C}}, [T(\vec{v}_2)]_{\mathcal{C}}, \ldots, [T(\vec{v}_k)]_{\mathcal{C}}$ forms a ___________.

First observe that each $[T(\vec{v}_j)]_{\mathcal{C}} \in$ ____ $(1 \leq j \leq k)$ since

$$=$$

NOTE: $R(L_A) = \{ \qquad \}$.

Suppose $\vec{y} \in R(L_A)$. Then

$$\vec{y} =$$

Choose $\vec{v} \in V$ such that $[\vec{v}]_{\mathcal{B}} = \vec{x}$.

Then $\vec{y} =$

$T(\vec{v}) \in$ _ _ _ so there are scalars $\alpha_1, \alpha_2, \ldots, \alpha_k$ such that p-3)

$T(\vec{v}) =$

Then

$\vec{y} =$

$=$

Hence the vectors

span $R(L_A)$.

We show that they are linearly independent.

Suppose

$= \vec{0}$

Then

$= \vec{0}$

so that

$= \vec{0}$

which implies

Since

Hence the vectors

form a basis for _ _ _. Therefore

$\operatorname{rank}(T) =$

Theorem 3.4 Let A be a $m \times n$ matrix and suppose P is an $m \times m$ invertible matrix, and Q is an $n \times n$ invertible matrix. (p.4)

Then

(a) $\text{rank}(AQ) =$

(b) $\text{rank}(PA) =$

(c) $\text{rank}(PAQ) =$

Proof:

(a) Suppose $\vec{y} \in R(L_{AQ})$. Then

Hence $R(L_{AQ}) \subset$ _ _ _ _ _.

Now suppose $\vec{z} \in R(L_A)$ ie $\vec{z} =$

Let $\vec{x_2} =$

Then $(AQ)\vec{x}_2 =$

and

$\vec{z} \in$ _ _ _ _ _

Therefore $R(L_A) \subset$ and

$R(L_A) =$

Hence

$\text{rank}(A) =$

$\square$

(b) We will show that $N(L_{PA}) =$. (P5)

Suppose $\vec{x} \in N(L_A)$. Then

Hence $L_{PA}(\vec{x}) =$,

and $\vec{x} \in$ ___. Therefore

$N(L_A) \subset$ ____.

Now suppose $\vec{x}_1 \in N(L_{PA})$; i.e.
Then

so that $\vec{x}_1 \in$ ___, Therefore
$N(L_{PA}) \subset$, and

$N(L_{PA}) =$.

PA, and A are both __ $\times$ __ matrices so that

L_{PA} : __ $\rightarrow$ __ , & L_A : __ $\rightarrow$ ___.

By the Dimension Theorem

=

=

nullity$(L_{PA}) =$

Since $N(L_{PA}) =$. Therefore ,
rank$(L_{PA}) =$, & rank$(PA) =$. $\square$

Corollary (p. 6)

Elementary row & column operations are __ __ __ preserving (ie they do ___ ___ __ __ __ _ _ of a matrix).

Definition

Let $A \in M_{m\times n}(F)$ & suppose

$$A = [\vec{a_1} | \vec{a_2} | \cdots | \vec{a_n}],$$

ie. $\vec{a_1}, \vec{a_2}, \ldots, \vec{a_n}$ are the __ __ __ _ _ of A.

The Column Space of A denoted by __ __ __ __ _ _ is defined by

Column Space(A) = ___ ___ __ __ ___ __ _ $\subset$ __ _ .

Lemma Let $A \in M_{m\times n}(F)$. Then

Column Space(A) = __ __ __ _ .

PROOF:

Let $\vec{y} \in R(L_A)$. Then $\vec{y} =$,

and

$\vec{y} =$

(by a previous theorem) so that

$\vec{y} \in$ __ __ _ _ _ .

Similarly, if $\vec{y} \in$ Column Space(A) it is straightforward to show that $\vec{y} \in$ _ _ _ _ and the result follows.

Corollary (Thm 3.5 in TEXT) (p.7)

Let $A \in M_{m \times n}(F)$.

(i) _ _ _ _ _ _ _ = dim ColumnSpace(A).

(ii) Rank(A) = maximum _ _ _ _ _ _ _ _ _ _ _ _ _ _ _

PROOF:

(i) Rank(A) =

(ii) ColumnSpace(A) =

A maximal set of . . . .

. . . . basis for ColumnSpace(A).

Hence

Rank(A) =

=

. □

Example Find the rank of $A = \begin{pmatrix} 1 & 6 & -3 & 2 \\ -4 & -17 & 4 & -2 \\ 6 & 20 & -3 & 8 \\ -3 & -4 & -7 & 6 \end{pmatrix}$.

(p. 6)

Recall (p.9)

<u>Theorem 2.5</u> Let $T: V \to W$ be linear with $\dim V = \dim W = n < \infty$.

T.F.A.E:

(i)

(ii)

(iii) $\operatorname{rank}(T) =$

<u>Corollary 2 (p.102)</u>

Let $A \in M_{m\times n}(F)$.

Then A is invertible iff - - - is invertible where . . . .

<u>Theorem</u>

Let $A \in M_{n\times n}(F)$.

A is invertible iff $\operatorname{rank}(A) =$ - - - - - .

<u>PROOF</u>

A is invertible iff - - - - - - - - - - - - .

L_A is invertible iff - - - - - - - - - - .

But $\operatorname{rank}(A) =$ - - - - - - - - (by definition)

Hence

A is invertible iff $\operatorname{rank}(A) =$ - - - - . □

Example Determine whether $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$ (p.1)

is invertible.

Theorem: Let $A \in M_{m \times n}(F)$ & suppose $\text{rank}(A) = r$.
Then

(i) $r \leq$ [redacted] _ & $r \leq$ [redacted] _ _ _

(ii) A can be transformed into a matrix of the form

$$D = \left(\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array} \right)$$

by _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _.

(iii) Further there are invertible matrices P, Q s.th.

$D =$ _ _ _ _ _ _ _.

PROOF in PART

(i) rank(A) =

also each column of A is in ___ and so Column Space of A is a subspace of ___.
Hence

rank(A) =

Example Transform $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$ into a matrix D as in the Theorem.

(p. 12)

<u>Proposition</u> Let $A \in M_{n\times n}(F)$. If A is invertible then A^t is ______, and

$$(A^t)^{-1} = \_\_\_\_ .$$

<u>PROOF</u> Suppose $A \in M_{n\times n}(F)$ & A is invertible. Then

$$AA^{-1} = I = A^{-1}A.$$

So $\quad (AA^{-1})^t = I^t = (A^{-1}A)^t$

But $\quad I^t = I,$

$$(AA^{-1})^t = (A^{-1})^t A^t, \ \&$$

$$(A^{-1}A)^t = \_\_\_\_\_\_ .$$

Hence $\quad A^t \, \_\_\_ = I = \_\_\_ \, A^t.$

Therefore A^t is invertible &

$$(A^t)^{-1} = \_\_\_ ,$$

by ______. $\square$

<u>Corollary of Theorem</u>

Let $A \in M_{m\times n}(F)$. Then

$$\operatorname{rank}(A^t) = \_\_\_\_\_\_ .$$

<u>PROOF:</u> By the theorem there are a ______ matrices P, Q such that

$$D = \_\_\_ = \left(\begin{array}{c|c} \quad & \quad \\ \hline \quad & \quad \end{array}\right)$$

where $r =$ ______. Then

$D^t =$ ___ ___ = ___ ___ ___ ___ = $\left(\begin{array}{c|c} & \\ \hline & \end{array}\right)$ (p-13)

since $I_r{}^t =$ ____.

$\operatorname{rank}(A^t) =$ ___ ___ ___ ___ (since ___ ___ ___ ___ ___ ___ ___ ___ ___ are ___ ___ ___ ___)

$= \dim \text{Column Space}(\quad)$

$=$

$=$ $\boxed{}$

<u>Definition</u> Let $A \in M_{m \times n}(F)$ with rows

$\vec{r_1} = (\ , \ , \ \cdots , \ ),$
$\vec{r_2} = (\ , \ , \ \cdots , \ ),$
$\vdots$
$\vec{r}_{\_} = (\ , \ , \ \cdots , \ ).$

Then

Row Space$(A) :=$ ___ ___ ___ ___ ___ ___ $\subset F^{\_}$.

Example $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$. (p. 164)

Find dim Row Space(A).

Theorem: Let $A \in M_{m \times n}(F)$. Then

rank(A) = __________ = __________.

PROOF:

rank(A) = __________ = ________

= __________ = __________.

$\square$

Theorem. Let $A \in M_{n \times n}(F)$.
A is invertible if and only if
A is a product of ______________.

PROOF

($\Rightarrow$) Suppose $A \in M_{n \times n}(F)$ & A is invertible.
Then

rank(A) = _____

and

(1.15)

$$D =$$

the _ _ _ _ _; in fact ca be shown that
P, Q are products of _ _ _ _ _ . . .
A

$$A^{-1} = \_ \_ \_ \_ = \_ \_ \_ .$$

is a product of _ _ _ _ _ _ _ since
_ _ _ _ _. are _ _ _ _ _ _ _ _ _ _ .

$\square$

Theorem Let $A \in M_{n\times n}(F)$ (1.16)
A is invertible if and only if

the matrix $(A \mid \_\_\_)$ can be transformed into a matrix of the form $(\_\_\_ \mid \_\_\_)$ by elementary ___ operations in which case

$$A^{-1} = \_\_\_\_.$$

PROOF:

$(\Leftarrow)$ Suppose A is invertible. Then A is a product of ____ ___ say

$$A = \_\_\_\_\_\_\_\_\_\_.$$

Then

$$A^{-1} = \_\_\_\_\_\_\_\_$$
$$= \_\_\_\_\_\_\_\_.$$

For $1 \leq j \leq k$, each ___ is an ____ and corresponds to an ______
Apply each of these _____ to the matrix $(A \mid I_n)$ ie to A & ___.
This corresponds to multiplying A & I_n in turn on the ___ be each matrix ___.

Hence $(A \mid I_n) \to (\_\_\_\_\_\_ \mid \_\_\_\_\_\_)$
$= (\_\_\_ \mid \_\_\_)$

(p.19)

$$= \left(\_\_\_ \mid \_\_\_ \right). \quad \square$$

($\Leftarrow$) EX.

<u>Example</u> Let $A = \begin{pmatrix} 1 & -2 & -2 \\ -3 & 7 & 6 \\ 2 & -1 & -3 \end{pmatrix} \in M_{3\times 3}(\mathbb{R})$

Show A is invertible & find A^{-1}!

$$\left(\begin{array}{ccc|ccc} 1 & -2 & -2 & 1 & 0 & 0 \\ -3 & 7 & 6 & 0 & 1 & 0 \\ 2 & -1 & -3 & 0 & 0 & 1 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|ccc} & & & & & \\ & & & & & \\ & & & & & \end{array}\right)$$

$$\to \left(\begin{array}{ccc|ccc} & & & & & \\ & & & & & \\ & & & & & \end{array}\right)$$

$$\to \left(\begin{array}{ccc|ccc} & & & & & \\ & & & & & \\ & & & & & \end{array}\right)$$

(p.68)

$\longrightarrow \left(\quad \middle| \quad \right)$

A is invertible since A ____________

Also

$$A^{-1} = \left(\quad \right)$$

Check: $AA^{-1} =$

Theorem Suppose V, W, Z are finite dimensional vector spaces, $T: V \to W$, $U: W \to Z$ are linear. Suppose A, B are matrices (over F) such that AB is defined.

Then

(1) rank(UT) ________

(2) rank(UT) ________

(3) rank(AB) ________

(4) rank(AB) ________

PROOF (1) Clearly $R(UT) \subset R(\quad)$. (p.19)

$\mathrm{rank}(UT) =$ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _.

(3) $\mathrm{rank}(AB) =$ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _

= _ _ _ _

(4) $\mathrm{rank}(AB) =$ _ _ _ _

= _ _ _ _ $\leq$ _ _ _ _

(2) Let $\mathcal{B}, \mathcal{C}, \mathcal{D}$ be ordered bases of V, W, Z respectively.

Let $A' = [T]$, $B' = [U]$

Then $C' = [UT] = \quad =$

$\mathrm{rank}(UT) = \quad =$

$\leq$ _ _ _

= _ _ _.

$\square$