Review

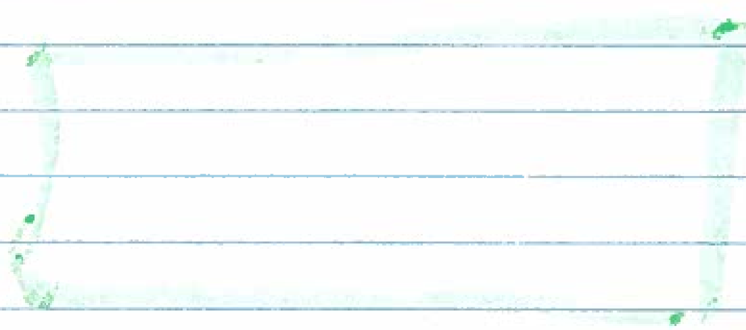
For Quiz 7

see

For EXAM 3

see

and



Corollary: Let $A \in M_{n\times n}(F)$ & suppose $\beta = [\vec{b_1}, \vec{b_2}, \ldots, \vec{b_n}]$ is an ordered basis of F^n.

Let $B = [\quad]_\beta$. Then

$$B = Q^{-1}AQ, \text{ where}$$

$$Q = [\qquad\qquad].$$

PROOF: The result follows from 2.23 since

$B =$

$A =$

and Q is . ☐

Definition: Let $A, B \in M_{n\times n}(F)$. We say B is similar to A if there is ____________ such that

$B =$

Example: $B = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ is similar to $A = [\quad]$.

<u>Proposition</u>

Let $A, B \in M_{n\times n}(F)$. If A and B are invertible, then AB is ________, and
$$(AB)^{-1} = \_\_\_\_\_\_\_\_.$$

PROOF: Suppose A, B are invertible $n\times n$ matrices. Then

$$AA^{-1} =\qquad\qquad ,\text{ and}$$
$$BB^{-1} =\qquad\qquad .$$

Now

$$(AB)(B^{-1}A^{-1}) =$$
$$=$$
$$=$$
$$=$$
$$=$$

and

$$(B^{-1}A^{-1})(AB) =$$
$$=$$
$$=$$
$$=$$
$$=$$

Hence $(AB)(\_\_\_) = (\_\_\_\_)(AB) = \_\_\_$.

Therefore AB is ________ and (by ______)

$$(AB)^{-1} = \qquad \square$$

$\boxed{\begin{array}{l}\underline{\text{Theorem}} \ (\text{Ex. 9 in } \S 2.4).\\ \text{Suppose } A, B \in M_{n\times n}(F) \text{ and } AB=C \text{ is } \_\_\ \_\_\_\_.\\ \text{Then } A \ \_\_\_ \ B \text{ are } \_\_\_\_\_\_\_\_.\end{array}}$

PROOF: Suppose A, B are $n \times n$ matrices &
$C = AB$ is invertible.

Consider $L_A : F^n \longrightarrow F^n$ & $L_B : F^n \longrightarrow F^n$

Suppose $\vec{x}, \vec{y} \in F^n$ & $L_B(\vec{x}) = L_B(\vec{y})$.
Then

$=$

$=$

$=$

which implies $=$ since C is $\_\_\_\_$
and hence L_C is $\_\_\_\_\_$ and $\_\_\_\_\_\_$.
Hence L_B is one-$\_\_\_\_$. Since
$L_B : V \xrightarrow{B} W$ where $V = W = \_\_\_$ this
implies L_B is $\_\_\_\_$ by Theorem 2.5.
Hence L_B is $\_\_\_\_\_$ and by Thm 2.18

$B =$ is invertible

where $\mathcal{E}$ is $\_\_\ \_\_\_\_\_\_\_\_$.
Next we show that L_A is $\_\_\_\_$.
Let $\vec{w}$ be any vector in F^n.
Let $\vec{y} = \_\_\_\_\_$. Then $\vec{y} \in \_\_\_$, and
$L_A(\vec{y}) =$

$=$

$=$

$=$

=

Hence L_A is ______. Since $L_A : V \to W$ where $V = W =$ ___ this implies that L_A is ______ by Theorem 2.5. Hence ~~L_A~~

$$A = \_\_\_\_\_\_$$

is invertible by Thm 2.18. $\square$

<u>Proposition</u> Let $A, B, C \in M_{n\times n}(F)$.
(1) If A is similar to B then __________.
(2) A is similar to ______.
(3) If A is similar to B and B is similar to C then __________.

<u>PROOF.</u>
(1) Suppose A is similar to B.
Then there's an invertible matrix Q such that

$$A = \_\_\_\_\_\_.$$

Then

$$QA =$$

$$QAQ^{-1} =$$

and

$$B = \_\_\_\_\_\_\_\_$$

since $P =$ is _____ and $P^{-1} =$ exists (clearly).

(2) A is similar to A since
I is invertible, $I^{-1} = I$ and
$A =$

(3) Suppose A is similar to B and B is similar to C. Then there are ______ matrices P and Q such that
$A =$ ______

and $B =$ ______.

$A =$
$=$

(by ______).
Hence
A is ______ ______ ______. $\square$

Chapter 3
Elementary Matrix Operations & Systems of Linear Equations

3.1 Elementary matrix operations & elementary matrices

Elementary Row Operations

(1) ________ two rows (_____)
(2) Multiply a row by __________ (___)
(3) Add ______________ (_____)

Example

$$A = \begin{pmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{pmatrix}$$

$$\xrightarrow{(\quad)} \begin{pmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ & & & & & \end{pmatrix}$$

$$\xrightarrow{(\quad)} \begin{pmatrix} & & & & & \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 1 & & & & & \\ 0 & & & & & \\ 0 & 3 & -6 & 6 & 4 & -5 \end{pmatrix}$$

Elementary Column Operations

(1) ________ two columns (____)
(2) Multiply a column by ________ (___)
(3) add ________________ (____)

Example

$$A = \begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix} \xrightarrow{(\quad)} \begin{pmatrix} 1 & 0 & 7 \\ 2 & & 8 \\ 3 & & 9 \end{pmatrix}$$

Definition

An elementary matrix is a ______ matrix which can be obtaining by performing ___ elementary ______ ___ operation on ___.

Example ①

(1)

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{-2R_1 + R_3} \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix} = E$$

E is a Type __ elementary matrix corresponding to the row operation ____.

NOTE

(1) $I \longrightarrow E$ also corresponds to the column operation ______

(2) $E \xrightarrow{\boxed{\quad\quad}} \underline{I}$

(3) $E^{-1} = \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix}$ is also an elementary matrix & it corresponds to the row operation _____.

Exercise: Check that $E^{-1}E = E^{-1}E = \underline{I}$.

Example (2)

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{(R_1 \leftrightarrow R_2)} E = \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix}$$

E is an elementary matrix of type ___ corresponding to the row operation ___. It also corresponds to the column operation ___.

$E^{-1} =$ ___ corresponds to ___

and $E \xrightarrow{\boxed{\quad\quad}} \underline{I}$.

Example (3)

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{(3R_2)} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix} = E$$

E is an elementary matrix of type ___ corresponding to the row operation ___. It also corresponds to the column operation ___.

$$E^{-1} = \begin{pmatrix} \quad & \quad & \quad \\ & & \\ & & \end{pmatrix}$$ which corresponds to _ _ _

and $E \xrightarrow{\boxed{\quad}} I$.

<u>Example</u>

Let $E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}$. E is the elementary 3×3 matrix corresponding to the row operation _____.

Let $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$

Then

$$EA = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix} \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

$$= \begin{pmatrix} \quad & \quad & \quad \\ & & \\ & & \end{pmatrix}$$

Observe that

$$A \longrightarrow \_\ \_\ \_$$

<u>Exercise</u> Show that $A \xrightarrow{(-2C_3+C_1)}$ _ _ _.

<u>Theorem</u> Let $A \in M_{m\times n}(F)$.

(1) Let E be the $m\times m$ elementary matrix corresponding to the elementary row operation $\mathcal{R}$. Then

$$A \xrightarrow{\mathcal{R}} \_\_\_.$$

(2) Let E' be the $\_\_ \times \_\_$ elementary matrix corresponding to the elementary column operation $\mathcal{C}$. Then

$$A \xrightarrow{\mathcal{C}} \_\_\_.$$

<u>Theorem</u> Elementary matrices are ___ ___, and the inverse is an ___ ___ ___ matrix of the ___ ______.

3.2 The Rank of a Matrix & Matrix Inverse

Definition Let $A \in M_{m\times n}(F)$. The rank of A (denoted by ________) is defined to be the ____.

In otherwords, $\text{rank}(A) =$ ____________.

Example Let $A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \in M_{2\times 2}(\mathbb{R})$.

Find rank(A).

$L_A : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ by $L_A(\vec{x}) = A\vec{x}$

Theorem 3.3 Let $T: V \to W$ be linear with V, W finite dimensional vector spaces with ordered bases β, $\mathcal{C}$ respectively. Let $A = [T]_\beta^{\mathcal{C}}$.
Then

$$\text{rank}(T) = \_\_\_\_\_\_\_\_.$$

PROOF:

Suppose $\{T(\vec{v}_1), T(\vec{v}_2), \ldots, T(\vec{v}_k)\}$ is ____________ where each $\vec{v}_j \in$ ____.

We will show that $[T(\vec{v}_1)]_{\mathcal{C}}, [T(\vec{v}_2)]_{\mathcal{C}}, \ldots, [T(\vec{v}_k)]_{\mathcal{C}}$ forms a ______________.

First observe that each $[T(\vec{v}_j)]_{\mathcal{C}} \in$ ____ $(1 \leq j \leq k)$ since

$$=$$

NOTE: $R(L_A) = \{ \qquad \}$.

Suppose $\vec{y} \in R(L_A)$. Then

$$\vec{y} =$$

Choose $\vec{v} \in V$ such that $[\vec{v}]_\beta = \vec{x}$.

Then $\vec{y} =$

p-3)

$T(\vec{v}) \in$ ___ so there are scalars $\alpha_1, \alpha_2, \ldots, \alpha_k$ such that

$T(\vec{v}) =$

Then

$\vec{y} =$

$=$

Hence the vectors

span $R(L_A)$.

We show that they are linearly independent.

Suppose

$= \vec{0}$

Then

$= \vec{0}$

so that

$= \vec{0}$

which implies

Since

Hence the vectors

form a basis for ___. Therefore

$\text{rank}(T) =$

P.4

<u>Theorem 3.4</u> Let A be a $m\times n$ matrix and suppose P is an $m\times m$ invertible matrix, and Q is an $n\times n$ invertible matrix.

Then

(a) $\text{rank}(AQ) =$

(b) $\text{rank}(PA) =$

(c) $\text{rank}(PAQ) =$

<u>Proof:</u>

(a) Suppose $\vec{y} \in R(L_{AQ})$. Then

Hence $R(L_{AQ}) \subset$ _____.

Now suppose $\vec{z} \in R(L_A)$ <u>ie</u> $\vec{z} =$

Let $\vec{x_2} =$.

Then $(AQ)\vec{x}_2 =$

and

$\vec{z} \in$ _____

Therefore $R(L_A) \subset$ and

$R(L_A) =$

Hence

$\text{rank}(A) =$ $\square$

(p.5)

(b) We will show that $N(L_{PA}) =$.

Suppose $\vec{x} \in N(L_A)$. Then

Hence $L_{PA}(\vec{x}) =$,

and $\vec{x} \in$ ___. Therefore
$N(L_A) \subset$ ___.

Now suppose $\vec{x}_1 \in N(L_{PA})$; i.e. .
Then

so that $\vec{x}_1 \in$ ___, Therefore
$N(L_{PA}) \subset$, and

$N(L_{PA}) =$.
PA, and A are both __ × __ matrices so that

L_{PA} : __ → __, & L_A : __ → ___.

By the Dimension Theorem

=

=

nullity $(L_{PA}) =$,
Since $N(L_{PA}) =$. Therefore
rank $(L_{PA}) =$, & rank$(PA) =$. □

(p. 6)

Corollary

Elementary row & column operations are ____ preserving (ie they do __________ of a matrix).

Definition

Let $A \in M_{m\times n}(F)$ & suppose

$$A = [\vec{a_1} | \vec{a_2} | \cdots | \vec{a_n}],$$

ie. $\vec{a_1}, \vec{a_2}, \ldots, \vec{a_n}$ are the _______ of A.

The Column Space of A denoted by _______ is defined by

Column Space(A) = ______________ $\subset$ ____.

Lemma Let $A \in M_{m\times n}(F)$. Then

Column Space(A) = _____.

PROOF:

Let $\vec{y} \in R(L_A)$. Then $\vec{y} =$,
and

$$\vec{y} =$$

(by a previous theorem) so that

$$\vec{y} \in \_\_\_\_\_\_.$$

Similarly, if $\vec{y} \in$ Column Space(A) it is straightforward to show that $\vec{y} \in$ ____ and the result follows.

Corollary (Thm 3.5 in TEXT)

Let $A \in M_{m\times n}(F)$.

(i) _ _ _ _ _ _ _ = dim ColumnSpace(A).

(ii) Rank(A) = maximum _ _ _ _ _ _ _ _ _ _ _ _ _ _ _

PROOF:

(i) Rank(A) =

(ii) ColumnSpace(A) =

a maximal set of . . . .

. . . . bases for ColumnSpace(A).

Hence

Rank(A) =

=

$\square$

Example Find the rank of $A = \begin{pmatrix} 1 & 6 & -3 & 2 \\ -4 & -17 & 4 & -2 \\ 6 & 20 & -3 & 8 \\ -3 & -4 & -7 & 6 \end{pmatrix}$.

(p. 5)

Recall

Theorem 2.5 Let $T: V \to W$ be linear with
$\dim V = \dim W = n < \infty$.

T.F.A.E.:

(i)

(ii)

(iii) $\operatorname{rank}(T) =$

Corollary 2 (p.102)

Let $A \in M_{m \times n}(F)$.
Then A is invertible iff ___ is invertible
where . . . .

Theorem

Let $A \in M_{m \times n}(F)$.
A is invertible iff $\operatorname{rank}(A) =$ . . . . . .

Proof

A is invertible iff . . . .
L_A is invertible iff . . . .
But $\operatorname{rank}(A) =$ . . . . (by definition)
Hence
A is invertible iff $\operatorname{rank}(A) =$ . . . . . $\square$

9.10)

Example Determine whether $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$ is invertible.

Theorem: Let $A \in M_{m\times n}(F)$ & suppose $\mathrm{rank}(A) = r$. Then

(i) $r \leq$ [redacted] __ & $r \leq$ [redacted] ___

(ii) A can be transformed into a matrix of the form

$$D = \left(\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array}\right)$$

by _ _ _ _ _ _ _ _ _ _ _ _ _ _ .

(iii) Further there are invertible matrices P, Q s.th

$D =$ _ _ _ _ _ _ .

PROOF in PART

(i) $\operatorname{rank}(A) =$

Also each column of A is in ___ and so Column Space of A is a subspace of ___.
Hence

$\operatorname{rank}(A) =$

Example Transform $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$ into a matrix D as in the Theorem.

Proposition Let $A \in M_{n\times n}(F)$. If A is invertible then A^t is ______, and $(A^t)^{-1} =$ ____.

PROOF Suppose $A \in M_{n\times n}(F)$ & A is invertible. Then

$$A A^{-1} = I = A^{-1} A,$$

so

$$(A A^{-1})^t = I^t = (A^{-1}A)^t$$

But

$$I^t = I,$$

$$(A A^{-1})^t = (A^{-1})^t A^t, \ \&$$

$$(A^{-1}A)^t = \text{\_\_\_\_\_\_}.$$

Hence $\quad A^t \text{\_\_\_} = I = \text{\_\_\_} A^t.$

Therefore A^t is invertible &

$$(A^t)^{-1} = \text{\_\_\_},$$

by ____. $\square$

Corollary of Theorem

Let $A \in M_{m\times n}(F)$. Then

$$\operatorname{rank}(A^t) = \text{\_\_\_\_\_}.$$

PROOF: By Theorem there are three ______ matrices P, Q such that

$$D = \text{\_\_\_\_} = \left(\begin{array}{c|c} & \\ \hline & \end{array}\right)$$

where $r =$ ______. Then

(p. 15)

$D^t =$ ______ $=$ ________ $= \left(\begin{array}{c|c} \quad & \quad \\ \hline \quad & \quad \end{array}\right)$

since $I_r^{\,t} =$ ____ .

$\operatorname{rank}(A^t) =$ ________ (since __________ ________ are ________)

$=$ dim Column Space(   )

$=$

$=$. $\square$

Definition Let $A \in M_{m \times n}(F)$ with rows

$$\vec{r_1} = (\ \ , \ \ , \ \ \cdots\cdots, \ \ ),$$
$$\vec{r_2} = (\ \ , \ \ , \ \ \cdots\cdots, \ \ ),$$
$$\vdots$$
$$\vec{r}_{\_\_} = (\ \ , \ \ , \ \ \cdots\cdots, \ \ ).$$

Then

Row Space(A) $:=$ ______________ $\subset F^{\_\_}$.

(p. 16)

Example $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$.

Find dim RowSpace(A).

Theorem: Let $A \in M_{m\times n}(F)$. Then

rank(A) = _________ = _________.

PROOF:

rank(A) = _________ = _________

= _________ = _________.

$\square$

Theorem. Let $A \in M_{n\times n}(F)$.

A is invertible if and only if

A is a product of ____________.

PROOF

($\Rightarrow$) Suppose $A \in M_{n\times n}(F)$ & A is invertible.

Then

rank(A) = ___

and

(1.15)

$$D =$$

the _ _ _ _ _; in fact can be shown that P, Q are products of _ _ _ _ _
/Δ

$$A^{-1} = \_\_\_\_ = \_\_\_.$$

is a product of _ _ _ _ _ _ _ since
_ _ _ _ _. are _ _ _ _ _ _ _ _ _ _ _ _ _. $\square$

(1.16)

<u>Theorem</u> Let $A \in M_{n \times n}(F)$

A is invertible if and only if

the matrix $(A \mid \_\_\_)$ can be transformed into a matrix of the form $(\_\_\_ \mid \_\_\_)$ by elementary ___ operations in which case

$$A^{-1} = \_\_\_.$$

PROOF:

$(\Leftarrow)$ Suppose A is invertible. Then A is a product of ______ say

$$A = \_\_\_\_\_\_\_\_.$$

Then

$$A^{-1} = \_\_\_\_\_\_\_\_$$

$$= \_\_\_\_\_\_\_\_.$$

For $1 \leq j \leq k$, each ___ is an ____ and corresponds to an ______

Apply each of these _____ to the matrix $(A \mid I_n)$ ie to A & ___.

This corresponds to multiplying A & I_n in turn on the ___ by each matrix ___.

Hence $(A \mid I_n) \to (\_\_\_\_\_ \mid \_\_\_\_\_)$

$$= (\_\_\_ \mid \_\_\_)$$

(p.17)

$$= \left(\_\_\_ \mid \_\_\_ \right). \quad \square$$

($\Leftarrow$) EX.

<u>Example</u> Let $A = \begin{pmatrix} 1 & -2 & -2 \\ -3 & 7 & 6 \\ 2 & -1 & -3 \end{pmatrix} \in M_{3\times3}(\mathbb{R})$

Show A is invertible & find A^{-1}!

$$\left(\begin{array}{ccc|ccc} 1 & -2 & -2 & 1 & 0 & 0 \\ -3 & 7 & 6 & 0 & 1 & 0 \\ 2 & -1 & -3 & 0 & 0 & 1 \end{array} \right)$$

$$\rightarrow \left(\begin{array}{ccc|ccc} & & & & & \\ & & & & & \\ & & & & & \end{array} \right)$$

$$\rightarrow \left(\begin{array}{ccc|ccc} & & & & & \\ & & & & & \\ & & & & & \end{array} \right)$$

$$\rightarrow \left(\begin{array}{ccc|ccc} & & & & & \\ & & & & & \\ & & & & & \end{array} \right)$$

(p.68)

$$\longrightarrow \left(\begin{array}{c|c} \quad\quad\quad & \quad\quad\quad \end{array}\right)$$

A is invertible since A ______________

Also

$$A^{-1} = \left(\quad\quad\quad\quad \right)$$

Check: $AA^{-1} =$

<u>Theorem</u> Suppose V, W, Z are finite dimensional vector spaces, $T: V \to W$, $U: W \to Z$ are linear. Suppose A, B are matrices (over F) such that AB is defined.

Then

(1) rank(UT) ________

(2) rank(UT) ________

(3) rank(AB) ________

(4) rank(AB) ________

PROOF (1) Clearly $R(UT) \subset R(\ )$. (p. 19)

$\operatorname{rank}(UT) = $ ______________________

(3) $\operatorname{rank}(AB) = $ ______________________

$= $ ____________

(4) $\operatorname{rank}(AB) = $ __________

$= $ __________ $\leq$ __________

(2) Let $\mathcal{B}, \mathcal{C}, \mathcal{D}$ be ordered bases of V, W, Z respectively.

Let $A' = [T]$, $B' = [U]$

Then $C' = [UT] = \qquad =$

$\operatorname{rank}(UT) = \qquad =$

$\leq$ __________

$=$ __________.

$\square$

(p.1)

3.3 Systems of Linear Equations – Theoretical Aspects

A system of linear equations in n unknowns (over a field F) can be written as

where $A \in$ ____ is the coefficient matrix

$\vec{x} =$ is the column vector of ____ &

$\vec{b} =$ is the column vector corresponding to ____ ____.

Example Write the system

$$(*) \quad \begin{cases} x_1 + 3x_2 = 4 \\ x_1 + 2x_2 = 5 \end{cases}$$

in matrix form and solve it.

Example The system

$$\begin{cases} x_1 + x_2 = 0 \\ x_1 + x_2 = 1 \end{cases}$$

has ___ solutions and can be written as

Example The system $\begin{cases} x_1 + x_2 = 0 \\ 2x_1 + 2x_2 = 0 \end{cases}$

has ________ solutions $\vec{x} =$

where ______. The system can be written as

Def^n

A linear system is homogeneous if it can be written in the form

$$\boxed{}$$

Otherwise it is ______________.

Note Any homogeneous system has at least one solution namely $\vec{x} =$ ___.

(8-3)

Theorem Let $A\vec{x} = \vec{0}$ be a homogeneous system of m equations in n unknowns over a field F.

Let $K = \{\vec{x} : A\vec{x} = \vec{0}\}$.

Then $K = N(\quad)$ and is a subspace of ___,

and $\dim K =$ _____.

Proof:

$$N(L_A) = \{ \qquad \}$$

$$= \{ \qquad \} = \quad .$$

$L_A :$ __ $\longrightarrow$ __ by $L_A(\vec{x}) =$ ___

is ______, so that $K = N(\quad)$ is a subspace of ___.

By The Dimension Theorem,

$$\dim N(\quad) = \text{\_\_\_\_\_},$$

and

$$\dim K = \text{\_\_\_\_\_}. \quad \square$$

(Ex 4)

Example

Find the dimension of the soln set of the system

$$(*)\begin{cases} x_1 + 2x_2 + x_3 + x_4 = 0 \\ \quad\;\; x_2 - x_3 + x_4 = 0 \end{cases}$$

and a basis.

$(*) \Leftrightarrow A\vec{x} = \vec{0}$ where $A = \begin{pmatrix} \quad \end{pmatrix}$, $\vec{x} = \begin{pmatrix} \quad \end{pmatrix}$.

Rank $(A) =$ dim RowSpace $(A) =$ ___

since A has ....

Hence (by the Theorem)

dim of Soln Set $=$

$A\vec{x} = \vec{0} \Leftrightarrow \Big\{$

where

$\Leftrightarrow \vec{x} =$

$=$

It is clear that $\left\{ \begin{pmatrix} \quad \end{pmatrix}, \begin{pmatrix} \quad \end{pmatrix} \right\}$ is a basis for the soln set.

NOTE Here $K =$

(p.5)

Corollary A homogeneous linear system with more unknowns than equations must have __________________

PROOF:

Let $m = \#$ of equations

$n = \#$ of ~~bottom~~ unknowns

for a homogeneous linear system

$$A\vec{x} = \vec{0}$$

where A is ______________ and ____

Let $K =$ Soln Set $= \{ \qquad\qquad \}$

$\dim K =$

Hence,

Theorem: Let $A \in M_{m\times n}(F)$, $\vec{b} \in F^m$.
Suppose $\vec{x} = \vec{s}$ is a particular soln to

(*) $$A\vec{x} = \vec{b}$$

Let K_H be the soln set of $A\vec{x} = \vec{0}$. (**)
Then the soln set of (*) is

$$K = \{ \qquad\qquad \}.$$

PROOF:

Suppose $\vec{x} \in K$ ie

Then

and $\vec{x} =$

Conversely, suppose

$\vec{x} =$

Hence the soln set of (*) is the set

$K = \{ \qquad \}$ □

Example Find the soln set of

$$(*) \begin{cases} x_1 + 2x_2 + x_3 + x_4 = 1 \\ \quad\quad x_2 - x_3 + x_4 = 1 \end{cases}$$

We know

$K_H = \{ \qquad \}$

A particular soln to (*) is $\vec{x} = \vec{s} = \left(\quad \right)$

Hence the general soln of $(*)$ is given by (p.7)

$$\vec{x} =$$

where

__Theorem__

Let $A\vec{x} = \vec{b}$

be a linear system. The system is consistent (i.e. has at least one soln) if & only if rank $(A) =$ ____ .

__PROOF__

The ~~~~~~~~~ system is consistent

iff

__Example__ Determine if $(*)\begin{cases} x_1 + x_2 - x_3 = 1 \\ 2x_1 + x_2 + 3x_3 = 2 \end{cases}$

has a soln.

3.4 Systems of Linear Equations – Computational Aspects

Definition Two linear systems are equivalent if ——————————.

Theorem Let $A \in M_{m\times n}(F)$, $C \in M_{m\times m}(F)$, & $\vec{b} \in F^m$. Suppose C is invertible. Then linear system

$$A\vec{x} = \vec{b} \iff$$

$$(\text{————————})$$

PROOF:

Let $K_1 = \{\vec{x} \in F^n : A\vec{x} = \vec{b}\}$,

$K_2 = \{\vec{x} \in F^n : (CA)\vec{x} = C\vec{b}\}$.

Suppose $\vec{x} \in K_1$. Then

$\square$

(p.2)

Corollary Let $A \in M_{n\times n}(F)$, $\vec{b} \in F^n$.
Suppose A is invertible. Then the linear system
$$A\vec{x} = \vec{b}$$
has the ______ solution $\vec{x} =$

PROOF:

Corollary Let $A\vec{x} = \vec{b}$
be a linear system (with m equations & n unknowns)
$$A\vec{x} = \vec{b} \iff A'\vec{x} = \vec{b}'$$
if $(A' \mid \vec{b}')$ can be ____ from $(\mid)$
by ______________________________.

PROOF Suppose $(A' \mid \vec{b}')$ can be ____ from $(\mid)$
by ______________________________
These ____ operations can be done by multiplying
on the ______ by the ______ matrices
$E_1, E_2, \dots, E_k$
Let $A = (\vec{a}_1 \mid \vec{a}_2 \mid \dots \mid \vec{a}_n)$.
Then

$$\left(A' \mid \vec{b}'\right) = \\ = \\ = \\ =$$

So that $A' =$ ___ & $\vec{b}' =$ ___,

Hence $A\vec{x} = \vec{b} \iff$

since

Definition: A matrix is said to be in reduced row echelon form if the following 3 conditions are satisfied.

(1) Any row containing a nonzero entry must ____ any row in which ________
ie all rows of zeros are ________.

(2) The first nonzero entry of each row is the ____________.

(3) The first nonzero entry of each row is ____ and occurs in a ________________

Reduced Row Echelon Form

$$\begin{bmatrix} \cdot & * & * & \cdot & \cdot & * & * & \cdot & * \\ & & & \cdot & \cdot & * & * & \cdot & * \\ & & & & \cdot & * & * & \cdot & * \\ & & & & & & & \cdot & * \\ & & & & & & & & \end{bmatrix}$$

(p.4)

Gaussian (or Gauss-Jordan) Elimination is the process in which a matrix is transforme by ____ ____ into ____.

Example Solve

$$\begin{cases} 2x_1 - 2x_2 - x_3 + 6x_4 - 2x_5 = 1 \\ x_1 - x_2 + x_3 + 2x_4 - x_5 = 2 \\ 4x_1 - 4x_2 + 5x_3 + 7x_4 - x_5 = 6 \end{cases}$$

$$\longleftrightarrow \left[\begin{array}{ccccc|c} & & & & & \\ & & & & & \\ & & & & & \end{array}\right] = A$$

$$\longleftrightarrow \left[\begin{array}{ccccc|c} & & & & & \\ 2 & 2 & -1 & 6 & -2 & 1 \\ 4 & -4 & 5 & 7 & -1 & 6 \end{array}\right]$$

$$\longleftrightarrow \left[\begin{array}{ccccc|c} 1 & -1 & 1 & 2 & -1 & 2 \\ 0 & & & & & \\ 0 & & & & & \end{array}\right]$$

$$\longleftrightarrow \left[\begin{array}{ccccc|c} 1 & -1 & 1 & 2 & -1 & 2 \\ 0 & 0 & 1 & & & \\ 0 & & & & & \end{array}\right]$$

(p.5)

$$\longleftrightarrow \left[\begin{array}{ccccc|c} 1 & -1 & 1 & 2 & -1 & 2 \\ 0 & 0 & 1 & & & \\ 0 & 0 & 0 & & & \end{array}\right]$$

$$\longleftrightarrow \left[\begin{array}{cccccc} 1 & -1 & 1 & 2 & -1 & 2 \\ 0 & 0 & 1 & & & \\ 0 & 0 & 0 & 1 & & \end{array}\right]$$

$$\longleftrightarrow \left[\begin{array}{ccccc|c} 1 & -1 & 1 & 0 & & \\ 0 & 0 & 1 & 0 & & \\ 0 & 0 & 0 & 1 & & \end{array}\right]$$

$$\longleftrightarrow \left[\begin{array}{ccccc|c} 1 & -1 & 0 & 0 & & \\ 0 & 0 & 1 & 0 & & \\ 0 & 0 & 0 & 1 & & \end{array}\right] = B$$

which is in ______________.

$x_1 =$

$x_3 =$

$x_5 =$

where

$\vec{x} =$ (p-6)

Theorem

The reduced row echelon form of a matrix is

Proposition

Elementary row operations do not change the ------------------------------.

Example

Let $A = [\vec{a}_1 \ \vec{a}_2 \ \vec{a}_3 \ \vec{a}_4 \ \vec{a}_5 \ \vec{a}_6]$

$B = [\vec{b}_1 \ \vec{b}_2 \ \vec{b}_3 \ \vec{b}_4 \ \vec{b}_5 \ \vec{b}_6]$

as in the previous example.

We see that

$$\vec{b}_6 =$$

Observe that

and $\vec{a}_6 =$

PROOF of Proposition

Let $A = [\vec{a}_1 | \vec{a}_2 | \cdots | \vec{a}_n]$

$B = [\vec{b}_1 | \vec{b}_2 | \cdots | \vec{b}_n]$

be $m \times n$ matrices in which B can be obtained from A by a single elementary row operation which corresponds to "multiplication on the ---- by

___ x _ _ _ _ _ _ _ _ _ matrix E.

Suppose the columns of B are related by

$$x_1 \vec{b}_1 + x_2 \vec{b}_2 + \dots + x_n \vec{b}_n = \vec{0}$$

where $x_1, x_2, \dots, x_n$ are _ _ _ _ _ _.

Hence

Corollary Let A be an $m \times n$ matrix.
Let B be the reduced row echelon form matrix of A.
A basis for the Column Space (A) is formed by
the _
_ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _.

Example: Let A, B be as before.
Columns _ _ _ _ _ _ _ _ _ _ _ _ are linearly independent
and form a _ _ _ _ _ set of linearly indept. columns.
It follows that the columns _ _ _ _ _ _ _ _ _ _ _ _ _ _
are linearly independent & form a _ _ _ _ _ set
of linearly indept columns of A.

Hence $\left\{ \qquad\qquad\qquad\qquad\qquad\qquad\qquad \right\}$

is a basis for Column Space (A).

Corollary Let B be the reduced echelon form of A. Then $\operatorname{rank}(A) =$ ______________________.

Example

Let W be the subspace of $M_{2\times2}(\mathbb{R})$ consisting of 2×2 symmetric matrices. We know $\dim W =$ ____.

Let

$$S = \left\{ \begin{pmatrix} 0 & -1 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 1 & 9 \end{pmatrix}, \begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix}, \begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix} \right\}.$$

Find a subset of S that is a basis of W.

We calculate the coordinate vector of each element of S relative to the standard basis of $M_{2\times2}(\mathbb{R})$:

$$S' = \left\{ \begin{bmatrix} 0 \\ -1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} \\ \\ \\ \\ \end{bmatrix}, \begin{bmatrix} \\ \\ \\ \\ \end{bmatrix}, \begin{bmatrix} \\ \\ \\ \\ \end{bmatrix} \right\}$$

Linear relationships between the elements of S correspond to ______________________ of S' and vice-versa.

(p.10)

Span (S') = Columnspace(A) where

$$A = \begin{bmatrix} 0 & & & & \\ -1 & & & & \\ 1 & & & & \\ 1 & & & & \end{bmatrix}.$$

The reduced row echelon form of A is

$$B = \begin{bmatrix} 1 & 0 & 3 & 0 & 4 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Hence the columns _ _ _ _ _ _ _ _ _ _ _ _ are linearly independent & form a basis for _ _ _ _.
It follows that $\left(\quad \right), \left(\quad \right), \left(\quad \right)$

form a basis for W (since

Example

Let $W = \{\vec{x} \in \mathbb{R}^4 : x_1 + x_2 + x_3 + x_4 = 0\} \subset \mathbb{R}^4$.

$S = \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \\ -6 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 5 \\ -9 \end{pmatrix} \right\}$ is a linearly independent subset of W.

Extend S to form a basis for W.

Determinants

(p.1)

There are several ways to define the determinant of a square matrix. In second semester calculus you would have encountered determinants of 2×2 & 3×3 matrices.

Notation: Determinant of A:

$$\det(A) = |A|.$$

$$\det\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} =$$

$$\det\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \quad a_{11}\ a_{22}\ a_{33}$$

$a_{11}\ a_{23}\ a_{32}$

$a_{12}\ a_{21}\ a_{33}$

$a_{1}\ a_{2}\ a_{3}$

$a_{1}\ a_{2}\ a_{3}$

$a_{1}\ a_{2}\ a_{3}$

Permutations of 1, 2, 3 :

	π		$sgn(\pi)$
1	2	3	

Diagonal Rule:

$$\begin{matrix} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{matrix}$$

Example Find $\det\begin{pmatrix} 2 & 3 & 5 \\ 1 & -1 & 2 \\ 3 & 4 & -1 \end{pmatrix}$

WARNING: The determinant of a 4×4 matrix can ______________.

Definition Let $n \geq 1$, $[n] = \{1, 2, 3, \ldots, n\}$. Then a permutation σ (in S_n) is a ________ and _____ map σ: ___ $\longrightarrow$ ___.

Example $\sigma : [3] \longrightarrow [3]$ by

n	$\sigma(n)$
1	
2	
3	

is a permutation.

There are ___ permutations (in S_3) of $[3]$.

Note: In general, there are ___ permutations (in S_n) of $[n]$.

Definition: The sign of the permutation $\sigma : [n] \to [n]$ is ___ where $N(\sigma)$ is the number of ________ such that ___ and ___.

Example: Let $\sigma : [3] \to [3]$ by

n	$\sigma(n)$
1	2
2	3
3	1

Find sign(σ).

Inversions:

$N(\sigma) =$

$\text{sign}(\sigma) =$

Definition: Let $A = (a_{ij})$ be an $n \times n$ matrix. The determinant of A (denoted by ___) is

$$\det(A) = \text{\_\_\_\_\_\_\_\_\_\_\_\_}$$

Co-factor Expansion

Let $A = (a_{ij})$ be an $n \times n$ matrix.
Let M_{ij} be the $(n-1) \times (n-1)$ matrix obtained from A by ------------------------
Then the cofactor expansion along the i-th row is

$$\det(A) = \text{-----------}$$

and cofactor expansion along the j-th column is

$$\det(A) = \text{-----------}.$$

Example Let

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 5 & 0 \\ 5 & -1 & -2 \end{pmatrix}$$

(i) Calculate $\det(A)$ using a cofactor expansion along the second row.
(ii) Calculate $\det(A)$ using a cofactor expansion along the third column.

Properties of the Determinant

(1) The effect on the determinant by a Type I operation (swap two rows or two columns) is to ______________________.

(2) The effect on the determinant of a Type II operation (multiply one row or column by a constant $c \neq 0$) is to ______________________.

(3) Type III row (or column) operations ______________________.

(4) The determinant of an upper (or lower) triangular matrix is ______________________.

Example Let

$$A = \begin{pmatrix} 1 & -1 & 2 & 3 \\ 2 & 2 & 0 & 2 \\ 4 & 1 & -1 & 1 \\ 1 & 2 & 3 & 0 \end{pmatrix}$$

Find $\det(A)$.

ALTERNATIVE APPROACH

Definition Let F be a field. A determinant function is a function $\det: M_{n\times n}(F) \longrightarrow F$ defined for each n satisfying the following:

(a) If A has two identical rows then
$$\det(A) = \_\_\_\_.$$

(b) If A^* is obtained from A by multiplying a single row of A by a scalar α, then
$$\det(A^*) = \_\_\_\_\_\_.$$

(c) If A, A^*, A^{**} are identical except possibly in their i-th row and if ________________________ then
$$\det(A^{**}) = \_\_\_\_\_\_\_\_.$$

Alternatively, write

$$\det(A) = \det(\vec{\alpha}_1, \vec{\alpha}_2, \ldots, \vec{\alpha}_n)$$

where the $\vec{\alpha}_j$ are the ___ of A. Then

$$\det(\vec{\alpha}_1, \vec{\alpha}_2, \ldots, \vec{\beta}_i + \vec{\delta}_i, \vec{\alpha}_{i+1}, \ldots, \vec{\alpha}_n)$$

$$= \ldots\ldots\ldots\ldots\ldots\ldots$$

(d) $\det(I_n) =$ ___.

Theorem 1

Suppose det is a determinant function & $A \in M_{n\times n}(F)$. Then

(i) If A has a row of zeros then $\det(A) =$ ___.

(ii) If A^* is obtained from A by interchanging two rows of A, then

$$\det(A^*) = \_\_\_\_.$$

(iii) If A^* is obtained from A by a Type III elementary row operation, then

$$\det(A^*) = \_\_\_\_\_.$$

(iv) If A is upper or lower triangular, then

$$\det(A) = \_\_\_\_\_\_\_\_.$$

Proof

(i) Let A^* be the matrix obtained from A by multiplying the row of zeros by zero.
Then $A =$ ___ , and by part ___ of the definition,

$$\det(A) = \_\_\_\_\_\_\_\_\_\_\_\_\_\_ .$$ □

(ii) Let $\vec{r_1}, \vec{r_2}, \dots, \vec{r_n}$ be the rows of A so that

$$A = (\vec{r_1}, \vec{r_2}, \dots, \vec{r_n}).$$

Let $1 \le i \ne j \le n$ & let A^* be obtained from A by swapping rows i & j, so that

$$A^* = (\vec{r_1}, \dots, \vec{r_j}, \dots, \vec{r_i}, \dots, \vec{r_n}).$$

$$\_\_\_ = \det(\vec{r_1}, \dots, \vec{r_i}+\vec{r_j}, \dots, \vec{r_i}+\vec{r_j}, \dots, \vec{r_n})$$
(by Defn ___)

$=$

$=$

$=$

$=$

Hence $\det(A^*) =$ ____ . □

(iii) Again, let

$$A = (\vec{r_1}, \vec{r_2}, \ldots, \vec{r_n}).$$

Suppose A^* is obtained from A by the Type III operation $\alpha R_i + R_j$. Then

$$A^* = (\vec{r_1}, \ldots, \vec{r}, \ldots, \qquad , \ldots, \vec{r_n})$$

By Defn ---

$\det(A^*) =$

$=$

$=$

(iv) Omit.

<u>Theorem 2</u> For each positive integer n, there is a unique determinant function.

<u>Theorem 3</u> Let $A, B \in M_{n\times n}(F)$.

(i) A is invertible if and only if ______.

(ii) $\det(AB) =$ ______

(iii) $\det(A^t) =$ ____.

<u>PROOF</u>

(i)($\Rightarrow$) Suppose A is invertible.

Then $\mathrm{rank}(A) = n$. It follows that the reduced row echelon form of A is ___.

(p.10)

Since every elementary row operation multiplies the determinant by a nonzero number,

$$\det(A) = \_\_\_\_\_\_\_\_\_\_$$

($\Leftarrow$) Suppose A is not invertible.

Then $\operatorname{rank}(A)$ _____.

It follows that the reduced row echelon form of A is a matrix A^* with a row of _______

Again since every elementary row operation multiplies the determinant by a _____, we have

$$\det(A) = \_\_\_\_\_\_\_\_\_\_\_\_$$

(ii) Omit Proof.

(iii) Let E be any elementary matrix. It can be shown that

$$\det(E^t) = \_\_\_\_\_$$

<u>Case 1</u> A is invertible.

Then A is a product of _____ matrices

$A = E_1 E_2 \cdots E_k$.

$A^t =$

$\det(A^t) =$

$=$

$=$

$=$

$=$

<u>Case 2</u> A is not invertible.
Then A^t is ______
since if A^t is _____ then $(A^t)^t =$ ___
is ______ which is a ______.
Hence $\det(A) =$ ___ and $\det(A^t) =$ ___,
and

$$\det(A^t) = \_\_\_.$$

$\square$

<u>Theorem 4</u>
If A is invertible then $\det(A^{-1}) = \frac{1}{\det(A)}$.

<u>Proof</u>:

Chapter 5 – Diagonalization

5.1 Eigenvalues & Eigenvectors

Definition Let V be a finite dimensional vector space over a field F so that $\dim V = n \geq 1$. A linear operator $T: V \longrightarrow V$ is *diagonalizable* if V has a ______________ such that

$$\begin{bmatrix} \quad \end{bmatrix} = \begin{bmatrix} \quad \end{bmatrix} = \begin{bmatrix} \quad & & \\ & \quad & \\ & & \quad \end{bmatrix}$$

is a ______________ matrix

Example Let $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ by $T(\vec{x}) = A\vec{x}$ where $A = \begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix}$.

Show that A is diagonalizable by finding $[T]_{\mathcal{B}}$ where $\mathcal{B} = \left[\vec{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \ \vec{v}_2 = \begin{pmatrix} 3 \\ 4 \end{pmatrix} \right]$.

NOTE:
In general
$$[T]_{\mathcal{B}} = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & & \vdots \\ 0 & 0 & \ddots & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_n \end{bmatrix}$$

where $\mathcal{B} = [\vec{v_1}, \vec{v_2}, \ldots, \vec{v_n}]$ if and only if

$$[T(\vec{v_j})]_{\mathcal{B}} = \qquad ,$$

iff

$$T(\vec{v_j}) = \_\_\_\_\_\_\_\_$$

for each $\leq j \leq n$.

Definition: Let $T: V \longrightarrow V$ be linear.
A vector $\vec{v}$ in V is an eigenvector of T
if ________________________ such that
______________.

Definition Let V be a vector space over a field F, &
let $T: V \longrightarrow V$ be linear. The scalar $\lambda \in F$
is called a eigenvalue of T if ______________
________ such that
______________.

Example Let $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ by $T(\vec{x}) = A\vec{x}$,
where $A = \begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix}$. Let $\vec{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$.

Then

$$T(\vec{v}_1) =$$

$$T(\vec{v}_2) =$$

So

Theorem Let V be a finite dimensional vector space over a field F & $\dim V = n \geq 1$. Then T is diagonalizable if and only if V has a _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _.

PROOF:

($\Rightarrow$) Suppose $T: V \to V$ is diagonalizable.

($\Leftarrow$) (EX.)

Definition: Let $A \in M_{n \times n}(F)$. We say $\vec{v} \in$ _ _ _ is an eigenvector of A if

_ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _.

<u>Definition</u> Let $A \in M_{n\times n}(F)$. We say the scalar $\lambda \in$ __ is an <u>eigenvalue</u> of A if

_ _ _ _ _ _ _ _ _ _ _ _ _

<u>Theorem</u> Let $A \in M_{n\times n}(F)$.
The scalar λ is an eigenvalue of A if and only if $\det($_ _ _$)$ _ _ _.

<u>PROOF.</u>

<u>Example</u> Find the eigenvalues of $A = \begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix}$ in $\mathbb{R}$.

Definition Let $A \in M_{n\times n}(F)$.
The polynomial $p(t) =$ ________
is called the characteristic polynomial of A.

NOTE In general, $p(t)$ is a polynomial of degree __
with coefficients in ______.

Example Find the characteristic polynomial of
$$A = \begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix} \in M_{2\times 2}(\mathbb{R}).$$

NOTE: The roots of the characteristic polynomial
$p(t) =$ ________. in the ______
are the __________ of A.

Proposition Let $A, B \in M_{n\times n}(F)$.
If A is similar to B then A and B
have the ______________.

PROOF: Suppose A is similar to B.

Corollary

If $A, B \in M_{n\times n}(F)$ are similar then A and B have the same __ __ __ __ __ and the same __ __ __ __ __ __ __ .

Proposition

Let $T: V \longrightarrow V$ be linear where V are finite dim. vector spaces over F where $\dim V = n \geq 1$.
Suppose $\mathcal{B}, \mathcal{C}$ are ordered bases of V.
Then the matrices $[T]_{\mathcal{B}}$ and $[T]_{\mathcal{C}}$ have the same __ __ __ __ __ __ __ __ .

PROOF:

Definition Suppose $1 \leq \dim V = n < \infty$ & $T: V \to V$ is linear. The characteristic polynomial of T is

$$p(t) =$$

where -

Example Let $T: P_2(\mathbb{R}) \longrightarrow P_2(\mathbb{R})$ by $T(p(x)) = x p'(x)$. Find the characteristic polynomial of T.

<u>Example</u> Let $A = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 2 & 0 \\ 0 & 1 & 3 \end{pmatrix} \in M_{3\times 3}(\mathbb{R})$.

Find the eigenvalues & eigenvectors of A.

(p. 10)

Example Let $A = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 2 & 0 \\ 0 & 1 & 3 \end{pmatrix}$.

Find a basis for $\mathbb{R}^3$ consisting of eigenvectors of A.
Find an invertible matrix Q such that
$$Q^{-1}AQ = D$$
is a diagonal matrix.

Theorem Let $A \in M_{n\times n}(F)$. There is a invertible matrix $Q \in M_{n\times n}(F)$ such that

$$D = Q^{-1}AQ$$

is a diagonal matrix if and only if A has _

Further If A has _ $\in F^n$ then

$$Q^{-1}AQ = D = \left(\quad \right)$$

is a diagonal matrix where $Q = [\quad]$.

PROOF.

$(\Longrightarrow)$

PROBLEM Let $1 \leq \dim V = n < \infty$ & $T: V \to V$ be linear.

Then.

(i) Find eigenvalues of T.

(ii) Find if possible a basis $\mathcal{B}$ for V such that $[T]_{\mathcal{B}}$ is a diagonal matrix.

Solution:

* Choose a basis for V (usually - - - - -) say $\mathcal{E}$ and calculate $[T]_{\mathcal{E}} = A$.

* Calculate det(_ _ _ _)
and find the _ _ _ _ _ _ of A.

* Find all the _ _ _ _ _ _ _ of A and choose if possible a set of _ _ _ _ _ _ _ _ _ _ with _ _ _ _ _ _ _ _ _ _ _ _ _ _.

* Let $\vec{v_1}, \vec{v_2}, \ldots, \vec{v_n} \in V$ where $[\vec{v_j}]_{\mathcal{E}} =$ - - - - for each j.
Then $\mathcal{B} = [\qquad\qquad]$ will be a basis for V &

$$[T]_{\mathcal{B}} = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}.$$

#4 (e) (p.257)

Let $T: P_2(\mathbb{R}) \to P_2(\mathbb{R})$ by $T(f(x)) = xf'(x) + f(2)x + f(3)$.

Find the eigenvalues of T and an ordered basis of $P_2(\mathbb{R})$ such that $[T]_\beta$ is a diagonal matrix.

(p.17)

(p.18)

<u>Theorem</u> Suppose V is a finite dimensional vector space over a field F where $\dim V = n \geq 1$, and $T: V \longrightarrow V$ is linear. Let $\mathcal{B}$ be an ordered basis of V & let

$$\phi_{\mathcal{B}} : V \longrightarrow \mathbb{F}^n \text{ by } \phi_{\mathcal{B}}(\vec{v}) = [\vec{v}]_{\mathcal{B}},$$

which is an isomorphism.

Let $A = [T]_{\mathcal{B}}$.

We have the following diagram

$$\begin{tikzcd} V \arrow[r] \arrow[d] & V \\ F^n \arrow[r] & F^n \arrow[u] \end{tikzcd}$$

Then

(i) λ is an eigenvalue of T
iff __________________.

(ii) $\vec{v}$ is an eigenvector of T iff ________
____

(iii) Suppose $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n \in F^n$ are linearly ________
__________ of A.
Let $\vec{v}_1 =$ _____, ..., $\vec{v}_n =$ _____.
Then
$\mathcal{C} = [\ldots\ldots\ldots]$ is a basis of V
consisting of __________ of T and

(p.29)

$[T]_{\mathcal{C}}$ is a _ _ _ _ _ _ _ matrix.

PROOF.

(i)

($\Rightarrow$) Suppose λ is an eigenvalue of T.

($\Leftarrow$) (Ex)

(ii) (Ex)

(iii) $\vec{x}_1 =$ . . . . . , $\vec{x}_2 =$ . . . . . , ... , $\vec{x}_n =$ . . . . .

are _ _ _ _ _ _ _ _ .

5.2 Diagonalizability

Recall, $T: V \to V$ linear ($\dim V = n$) is diagonalizable iff V has a basis of ______________________.

Theorem Suppose $T: V \to V$ is linear & $\vec{v_1}, \vec{v_2}, \ldots, \vec{v_k}$ are eigenvectors of T with __________ eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$ respectively. Then the vectors $\vec{v_1}, \vec{v_2}, \ldots, \vec{v_k}$ are linearly ___________.

Proof: We proceed by induction on k.

The result is true for $k=1$ since if $\vec{v_1}$ is ______________. Then ______ and so $\{\vec{v_1}\}$ is ______________.

Now assume the result is true for $k=m$ and assume $\vec{v_1}, \vec{v_2}, \ldots, \vec{v_{m+1}}$ are eigenvectors of T with corresponding eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_{m+1}$ (respectively) and the λ_j are distinct ie ______________.

We want to show that ______________ are ______________.

So assume

$$c_1\vec{v_1} + c_2\vec{v_2} + \cdots + c_m\vec{v_m} + c_{m+1}\vec{v_{m+1}} = \vec{0}$$

where $c_1, c_2, \ldots, c_m, c_{m+1} \in F$. Then

(p.2)

Corollary Suppose $1 \leq \dim V = n < \infty$ & $T: V \to V$ is linear. If T has n ______ eigenvalues then T is __________.

PROOF:

Example. Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ by $T(\vec{x}) = A\vec{x}$ where $A = \begin{pmatrix} 4 & 2 \\ 7 & 1 \end{pmatrix}$ so that $T = L_A$.

NOTE:

* The converse of the Theorem is ________.

* The converse of the Corollary is ______.

For example (see Section 5.1).

Let $A =$

Let $A = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 2 & 0 \\ 0 & 1 & 3 \end{pmatrix}$.

We saw that has only _ _ _ _ _ eigenvalues ($\lambda =$ _ _ _ _ _) but we were able to find _ _ _ _ linearly independent eigenvectors.

So L_A is _ _ _ _ _ _ _. even though it has only _ _ _ _ _ _ _ eigenvalues.

<u>Definition</u> Let $f(t) \in P(F)$ (i.e. $f(t)$ is a polynomial with coefficients in the field F).
We say $f(t)$ <u>splits over F</u> if

_ _.

<u>Example</u>

(1) The polynomial $t^2 - 1$ _ _ _ over $\mathbb{R}$ since

(2) The polynomial $t^2 + 1$ _ _ _ _ _ _ _ over $\mathbb{R}$
But it _ _ _ _ _ _ over $\mathbb{C}$ since

<u>Theorem</u> Let V be a finite-dimensional vector over F where $1 \leq \dim V = n$. If $T: V \to V$ is linear and diagonalizable then the characteristic polynomial of T _ _ _ _ _ _ _ _.

(p.5)

proof:

Corollary Let V be a finite dimensional vector space over a field F with $1 \le \dim V = n$. If the characteristic polynomial of linear transformation $T: V \to V$ does __ split over __ then __________________ __.

Example Explain why $T: \mathbb{R}^2 \to \mathbb{R}^2$ by $T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 - 2x_2 \\ x_1 - x_2 \end{pmatrix}$ is not diagonalizable.

NOTE: $T: \mathbb{C}^2 \longrightarrow \mathbb{C}^2$ by $T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 - 2x_2 \\ x_1 - x_2 \end{pmatrix}$

is diagonalizable since

Ex Find two linearly independent eigenvectors of T.

Definition. Let λ be an eigenvalue of a linear operator $T: V \to V$. k is the <u>algebraic multiplicity</u> of λ if k is the ____

Example Let $T: \mathbb{R}^3 \to \mathbb{R}^3$ by $T(\vec{x}) = A\vec{x}$ where

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 2 & 0 \\ 0 & 1 & 3 \end{pmatrix}.$$

Find the algebraic multiplicity of each eigenvalue.

<u>Definition</u> Let λ be an eigenvalue of a linear operator $T: V \longrightarrow V$. The set

$$\mathcal{E}_\lambda := \{ \qquad \qquad \} = \{ \qquad \qquad \}$$

$$= N(\qquad \qquad)$$

is called the <u>eigenspace of T</u> corresponding to the eigenvalue λ.

___ ___ ___ ___ is called the <u>geometric multiplicity</u> of λ.

<u>Theorem</u> Suppose $1 \leqslant \dim V = n$ & $T: V \to V$ is linear. If λ is an eigenvalue of T then

$$1 \leqslant \_\_\_\_\_\_\_\_ \leqslant \_\_\_\_\_\_\_\_\_\_\_\_.$$

SPRING 2013 – MAS 4105 — LINEAR ALGEBRA — EXAM 3

Friday, April 19, 2013

NAME: ________________________________

INSTRUCTIONS:
- Write in complete sentences.
- Proofs should be written in a proper and coherent manner.
- Show all necessary work and explanations.

TOTAL POSSIBLE: 50 points.
DO ONLY FOUR QUESTIONS.

1. $[1 + 1.5 + (2 + 2 + 2 + 2 + 2) = 12.5$ pts$]$
Let $\mathcal{B}$ and $\mathcal{C}$ be ordered bases of a finite dimensional vector space V over a field F.

(i) Define the **change of coordinates matrix** from $\mathcal{B}$ to $\mathcal{C}$.

(ii) Complete:

Theorem: Suppose $T : V \longrightarrow V$ is linear and Q is the change of basis matrix from ________ to ________. Then

$$[T]_{\mathcal{B}} = \underline{\hspace{10cm}}.$$

(iii) Let $T : P_1(\mathbb{R}) \longrightarrow P_1(\mathbb{R})$ by

$$T(a + bx) = (a + 2b) + (2a + b)x.$$

You are given that T is linear. Let

$$\mathcal{B} = [1 + x, 1 - x], \qquad \mathcal{C} = [1, x].$$

(a) Find the change of coordinates matrix from $\mathcal{B}$ to $\mathcal{C}$.

(b) Find the change of coordinates matrix from $\mathcal{C}$ to $\mathcal{B}$.

(c) Find $[T]_{\mathcal{B}}$.

(d) Find $[T]_{\mathcal{C}}$.

(e) Find an invertible matrix Q such that

$$[T]_{\mathcal{B}} = Q^{-1}[T]_{\mathcal{C}}Q.$$

2. [1 + 6 + 5.5 = 12.5 pts]

(i) Let $A \in M_{m\times n}(F)$. Define the **rank** of A.

(ii) Let A be an $m \times n$ matrix and B be an $n \times p$ matrix.

Prove that $N(L_B) \subset N(L_{AB})$. Hence using the Dimension Theorem (or otherwise) prove that $\operatorname{rank}(AB) \leq \operatorname{rank}(B)$. Remember that $N(T)$ denotes the Null Space of T.

(iii) Let

$$P = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix} \in M_{3\times 3}(\mathbb{R}).$$

Explain why P is invertible and find P^{-1}.

3. [2 + 2.5 + 3 + (2 + 3) = 12.5 pts] Let

$$\vec{v}_1 = \begin{pmatrix} 1 \\ -3 \\ -3 \\ 2 \\ 2 \\ 2 \\ 3 \end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix} 3 \\ -9 \\ -9 \\ 6 \\ 6 \\ 6 \\ 9 \end{pmatrix}, \quad \vec{v}_3 = \begin{pmatrix} 5 \\ -14 \\ -21 \\ 10 \\ 6 \\ 17 \\ 9 \end{pmatrix}, \quad \vec{v}_4 = \begin{pmatrix} 40 \\ -113 \\ -162 \\ 80 \\ 52 \\ 129 \\ 78 \end{pmatrix}, \quad \vec{v}_5 = \begin{pmatrix} 6 \\ -12 \\ -53 \\ 13 \\ -15 \\ 55 \\ -19 \end{pmatrix},$$

and let A be the matrix whose columns are the vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4, \vec{v}_5$. You are given that the reduced echelon form matrix of A is the matrix

$$R = \begin{pmatrix} 1 & 3 & 0 & 5 & 0 \\ 0 & 0 & 1 & 7 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Suppose

$$W = \operatorname{Span}(\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4, \vec{v}_5).$$

Then W is a subspace of $\mathbb{R}^7$.

(i) What is $\dim W$?

(ii) Find a basis for W.

(iii) Write the vector $\vec{v}_4$ as a linear combination of $\vec{v}_1$ and $\vec{v}_3$.

(iv) Let

$$V = \{\vec{x} \in \mathbb{R}^5 \; : \; A\vec{x} = \vec{0}\},$$

where as above $A = [\vec{v}_1 \,|\, \vec{v}_2 \,|\, \vec{v}_3 \,|\, \vec{v}_4 \,|\, \vec{v}_5]$.

(a) What is $\dim V$?

(b) Find a basis for V.

4. $[3 + 5 + 4.5 = 12.5 \text{ pts}]$

(i) Let F be a field. Define what it means to say that $\det : M_{n\times n}(F) \longrightarrow F$ is a **determinant function**.

(ii) Suppose det is a determinant function and A is an $n \times n$ matrix. Suppose A^* is obtained from A by interchanging two rows of A. Prove that

$$\det(A^*) = -\det(A),$$

using the definition of a determinant function.

Hint: Let $\vec{r}_1, \vec{r}_2, \ldots \vec{r}_n$ be the rows of A so that $A = (\vec{r}_1, \ldots, \vec{r}_i, \ldots, \vec{r}_j, \ldots \vec{r}_n)$, and let $A^* = (\vec{r}_1, \ldots, \vec{r}_j, \ldots, \vec{r}_i, \ldots \vec{r}_n)$. Use the fact that

$$0 = \det(\vec{r}_1, \ldots, \vec{r}_i + \vec{r}_j, \ldots, \vec{r}_i + \vec{r}_j, \ldots \vec{r}_n).$$

(iii) Let $A, B \in M_{n\times n}(\mathbb{R})$ be such that $AB = -BA$. Prove that if n is odd then either A or B is not invertible.

5. $[1 + 1 + 2 + 3 + (0.5 + 3 + 2) = 12.5 \text{ pts}]$

(i) Define what it means for a linear operator T on a finite-dimensional vector space V to be **diagonalizable**.

(ii) Let T be a linear operator on a finite-dimensional vector space V. Define what it means for a vector $\vec{v} \in V$ to be an **eigenvector** of T.

(iii) Let $A \in M_{n \times n}(F)$. Complete:

The scalar $\lambda \in F$ is an eigenvalue of A if and only if

det(____________________) = ______.

(iv) Let $A \in M_{n \times n}(F)$. Prove that A and A^t have the same eigenvalues.

(v) Let

$$A = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ -1 & 0 & 3 & 1 \\ 0 & 0 & 0 & 3 \end{pmatrix} \in M_{4 \times 4}(\mathbb{R}).$$

You are given that the characteristic polynomial of A is

$$p(t) = (t-2)^2(t-3)^2.$$

(a) Find the eigenvalues of the matrix A.

(b) Find the dimension of each eigenspace.

(c) Determine whether A is diagonalizable.

Frank Garvan (http://people.clas.uf

Department *of* Mathematics

Chapter 3 – Elementary Matrix Equations

3.1 Elementary Matrix Operations and Elementary Matrices (p.151):
1, 2, 3

3.2 The Rank of a Matrix and Matrix Inverses (pp.165-168):
1, 2, 3, 4, 5, 6abc, 7, 8, 17, 19, 21*

3.3 Systems of Linear Equations – Theoretical Aspects (pp.179-181):
1, 2, 3, 4, 5, 7, 8, 10

3.4 Systems of Linear Equations – Computational Aspects (pp.195-198
1, 2, 4, 5, 6, 7, 8, 10, 11, 12

Frank Garvan (http://people.clas.uf

Department *of* Mathematics

Chapter 4 – Determinants

4.2 Determinants of Order n (pp.220-222):
1, 2, 3, 5, 7, 9, 13, 21, 25, 26

4.3 Properties of Determinants (pp.228-229):
1a-f, 10, 11, 12, 15, 17

NOTE: Also redo examples and proofs in the ONLINE REVIEW NOT

Frank Garvan (http://people.clas.uf

Department *of* Mathematics

Chapter 5 – Diagonalization

5.1 Eigenvalues and Eigenvectors (pp.256-259):
2, 3, 6, 8, 9, 12, 14, 15, 17

5.2 Diagonalizability (pp.279-280):
1(a)-(g), 2, 3, 7, 8, 9, 10, 12, 13