Chapter 5 — Diagonalization

5.1 Eigenvalues & Eigenvectors

Definition Let V be a finite dimensional vector space over a field F so that $\dim V = n \geq 1$. A linear operator $T: V \longrightarrow V$ is diagonalizable if V has an ordered basis $\mathcal{B}$ such that

$$[T]_{\mathcal{B}} = [T]_{\mathcal{B}}^{\mathcal{B}} = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_n \end{bmatrix}$$

is a diagonal matrix

Example Let $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ by $T(\vec{x}) = A\vec{x}$ where $A = \begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix}$.

Show that T is diagonalizable by finding $[T]_{\mathcal{B}}$ where $\mathcal{B} = \left[\vec{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \vec{v}_2 = \begin{pmatrix} 3 \\ 4 \end{pmatrix} \right]$.

$$T(\vec{v}_1) = A\vec{v}_1 = \begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \end{pmatrix}$$

$$= 2\begin{pmatrix} +1 \\ -1 \end{pmatrix} + 0\begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

$$= -2\vec{v}_1 + 0\vec{v}_2 .$$

$$T(\vec{v_2}) = \begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix}\begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 15 \\ 20 \end{pmatrix}$$

$$= 5\begin{pmatrix} 3 \\ 4 \end{pmatrix} = 0\begin{pmatrix} 1 \\ -1 \end{pmatrix} + 5\begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

$$= 0\,\vec{v_1} + 5\,\vec{v_2}.$$

Thus

$$[T]_{\mathcal{B}} = [T]_{\mathcal{B}}^{\mathcal{B}} = \Big[\, [T(\vec{v_1})]_{\mathcal{B}} \quad [T(\vec{v_2})]_{\mathcal{B}} \,\Big]$$

$$= \begin{bmatrix} -2 & 0 \\ 0 & 5 \end{bmatrix}.$$ Hence T is diagonalizable.

NOTE:

In general
$$[T]_{\mathcal{B}} = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & & \vdots \\ 0 & 0 & \ddots & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_n \end{bmatrix}$$

where $\mathcal{B} = [\vec{v_1}, \vec{v_2}, \ldots, \vec{v_n}]$ if and only if

$$[T(\vec{v_j})]_{\mathcal{B}} = \lambda_j\, \vec{e_j} \quad \text{for each } j,$$

iff

$$T(\vec{v_j}) = \lambda_j\, \vec{v_j}$$

for each $1 \le j \le n$.

Definition: Let $T: V \longrightarrow V$ be linear.
A vector $\vec{v}$ in V is an eigenvector of T
if $\vec{v} \neq \vec{0}$ & there is a scalar λ such that

$$T(\vec{v}) = \lambda \vec{v}.$$

Definition Let V be a vector space over a field F, & let $T: V \longrightarrow V$ be linear. The scalar $\lambda \in F$ is called a eigenvalue of T if there is a vector $\vec{v}$ in V $\vec{v} \neq \vec{0}$ such that

$$T(\vec{v}) = \lambda \vec{v}.$$

Example Let $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ by $T(\vec{x}) = A\vec{x}$, where $A = \begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix}$. Let $\vec{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$.

Then

$$T(\vec{v}_1) = -2\vec{v}_1$$

$$T(\vec{v}_2) = 5\vec{v}_2.$$

So $\vec{v}_1$ = is an eigenvector of T with eigenvalue $\lambda = -2$,

$\vec{v}_2$ is a eigenvector of T with eigenvalue $\lambda = 5$.

The scalars $\lambda = -2, 5$ are eigenvalues of T.

<u>Theorem</u> Let V be a finite dimensional vector space over a field F & dim $V = n \geq 1$. Then T is diagonalizable if and only if V has a basis consisting of eigenvectors of T.

<u>PROOF.</u>

($\Longrightarrow$) Suppose $T: V \to V$ is diagonalizable. Then there is a basis $\mathcal{B} = [\vec{v_1}, \vec{v_2}, \ldots, \vec{v_n}]$ such that

$$[T]_{\mathcal{B}} = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}$$

for some scalars $\lambda_1, \lambda_2, \ldots, \lambda_n$. This implies

$T(\vec{v_j}) = \lambda_j \vec{v_j}$ for each $1 \leq j \leq n$,

& each $\vec{v_j} \neq \vec{0}$ since $\mathcal{B}$ is a basis.

So each $\vec{v_j}$ is a eigenvector of T, &

V has a basis consisting of eigenvectors of T.

($\Longleftarrow$) (<u>EX.</u>)

<u>Definition</u>: Let $A \in M_{n \times n}(F)$. We say $\vec{v} \in F^n$ is an <u>eigenvector of A</u> if $\vec{v} \neq \vec{0}$ & $A\vec{v} = \lambda \vec{v}$ for some scalar $\lambda \in F$.

Definition Let $A \in M_{n\times n}(F)$. We say the scalar $\lambda \in F$ is an eigenvalue of A if $A\vec{v} = \lambda\vec{v}$ for some vector $\vec{v} \in F^n$, $\vec{v} \neq \vec{0}$.

Theorem Let $A \in M_{n\times n}(F)$.
The scalar λ is an eigenvalue of A if and only if $\det(A - \lambda I) = 0$.

PROOF. λ is an eigenvalue of A iff $A\vec{v} = \lambda\vec{v}$ for some $\vec{v} \in F^n$, $\vec{v} \neq \vec{0}$.

$\iff$ $A\vec{v} - \lambda\vec{v} = \vec{0}$ for some $\vec{v} \neq \vec{0}$

$\iff$ $(A - \lambda I)\vec{v} = \vec{0}$ for some $\vec{v} \neq \vec{0}$

$\iff$ $N(T) \neq \{\vec{0}\}$ where $T: F^n \to F^n$ by $T(\vec{x}) = (A - \lambda I)\vec{x}$.

$\iff$ T is not one-to-one

$\iff$ $\operatorname{rank}(T) \neq \dim F^n = n$

$\iff$ T is not invertible

$\iff$ $[T]_{\mathcal{E}} = A - \lambda I$ is not invertible where $\mathcal{E}$ is the standard basis of F^n

$\iff$ $\det(A - \lambda I) = 0$.

$\square$

Example Find the eigenvalues of $A = \begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix}$ in $\mathbb{R}$.

$$\det(A - \lambda I) = \det\begin{pmatrix} 1-\lambda & 3 \\ 4 & 2-\lambda \end{pmatrix} = (1-\lambda)(2-\lambda) - 12$$
$$= \lambda^2 - 3\lambda - 10$$
$$= (\lambda - 5)(\lambda + 2) = 0$$

when $\lambda = -2, 5$. The eigenvalues of A are $\lambda = -2, 5$.

Definition Let $A \in M_{n\times n}(F)$.
The polynomial $p(t) = \det(A - tI)$
is called the characteristic polynomial of A.

NOTE In general, $p(t)$ is a polynomial of degree n
with coefficients in F.

Example Find the characteristic polynomial of
$$A = \begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix} \in M_{2\times 2}(\mathbb{R}).$$

$$p(t) = \det(A - tI) = \det\begin{pmatrix} 1-t & 3 \\ 4 & 2-t \end{pmatrix}$$
$$= (1-t)(2-t) - 12 = t^2 - 3t - 10$$
is the characteristic polynomial.

NOTE: The roots of the characteristic polynomial
$p(t) = \det(A - tI)$ in the field F
are the eigenvalues of A.

Proposition Let $A, B \in M_{n\times n}(F)$.
If A is similar to B then A and B
have the matrices A, B have the same characteristic polynomial.

PROOF: Suppose A is similar to B. Then there is
an invertible matrix $Q \in M_{n\times n}(F)$ such that
$$A = Q^{-1} B Q$$

$$\det(A - tI) = \det(Q^{-1}BQ - tI)$$
$$= \det(Q^{-1}BQ - tQ^{-1}Q) = \det(Q^{-1}BQ - Q^{-1}(tI)Q)$$
$$= \det(Q^{-1}(B - tI)Q)$$
$$= \det(Q^{-1})\det(B - tI)\det(Q)$$
$$= \det(Q^{-1})\det(Q)\det(B - tI)$$
$$= \det(B - tI)$$

Since Q is invertible $\det(Q^{-1}) = \frac{1}{\det(Q)}$.

Thus

$$\det(A - tI) = \det(B - tI),$$

and A, B have the same characteristic polynomial.

Corollary

If $A, B \in M_{n\times n}(F)$ are similar then A and B have the same eigenvalues and the same determinant.

Proposition

Let $T: V \longrightarrow V$ be linear where V are finite dim. vector spaces over F where $\dim V = n \geq 1$.
Suppose $\mathcal{B}, \mathcal{C}$ are ordered bases of V.
Then the matrices $[T]_{\mathcal{B}}$ and $[T]_{\mathcal{C}}$ have the same characteristic polynomial.

PROOF:

$$\begin{tikzcd}[\vec{v}]_{\mathcal{B}} \arrow[r, "{[T]_{\mathcal{B}}}"] \arrow[d, "Q"'] & [T(\vec{v})]_{\mathcal{B}} \\ [\vec{v}]_{\mathcal{C}} \arrow[r, "{[T]_{\mathcal{C}}}"] & [T(\vec{v})]_{\mathcal{C}} \arrow[u, "Q^{-1}"']\end{tikzcd}$$

where Q is the change of basis matrix from $\mathcal{B}$ to $\mathcal{C}$. Then

$$[T]_{\mathcal{B}} = Q^{-1}\,[T]_{\mathcal{C}}\,Q.$$

Thus the matrices $[T]_{\mathcal{B}}$, $[T]_{\mathcal{C}}$ are similar & have the same characteristic polynomial.

<u>Definition</u> Suppose $1 \leq \dim V = n < \infty$ & $T: V \to V$ is linear. The <u>characteristic polynomial</u> of T is

$$p(t) = \det(A - tI)$$

where $A = [T]_{\mathcal{B}}$ & $\mathcal{B}$ is any ordered basis of V.

<u>Example</u> Let $T: P_2(\mathbb{R}) \to P_2(\mathbb{R})$ by $T(p(x)) = x p'(x)$. Find the characteristic polynomial of T.

- $\mathcal{B} = [1, x, x^2]$ is an ordered basis of $P_2(\mathbb{R})$.

$$T(1) = x \cdot 0 = 0 = 0 + 0x + 0x^2$$
$$T(x) = x \cdot 1 = x = 0 + 1x + 0x^2$$
$$T(x^2) = x \cdot (2x) = 2x^2 = 0 + 0x + 2x^2$$

So $A = [T]_{\mathcal{B}} = \left[[T(1)]_{\mathcal{B}} \;\; [T(x)]_{\mathcal{B}} \;\; [T(x^2)]_{\mathcal{B}} \right]$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$p(t) = \det(A - tI) = \det \begin{pmatrix} -t & 0 & 0 \\ 0 & 1-t & 0 \\ 0 & 0 & 2-t \end{pmatrix} \quad (\text{upper } \Delta)$$

$$= -t(1-t)(2-t) = -t(t-1)(t-2)$$

is the characteristic polynomial of T.

<u>Example</u> Let $A = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 2 & 0 \\ 0 & 1 & 3 \end{pmatrix} \in M_{3\times 3}(\mathbb{R})$.

Find the eigenvalues & eigenvectors of A.

$$\det(A - \lambda I) = \det \begin{pmatrix} 2-\lambda & 1 & 1 \\ 0 & 2-\lambda & 0 \\ 0 & 1 & 3-\lambda \end{pmatrix}$$

$$= (2-\lambda)\det\begin{pmatrix} 2-\lambda & 0 \\ 1 & 3-\lambda \end{pmatrix} \quad (\text{by cofactor exp along first column})$$

$$= (2-\lambda)(2-\lambda)(3-\lambda) = (2-\lambda)^2(3-\lambda) = 0$$

iff $\lambda = 2, 3$. The eigenvalues of A are $\lambda = 2, 3$.

$\boxed{\lambda = 3}$ We solve $(A - 3I)\vec{x} = \vec{0}$

$$\Leftrightarrow \left(\begin{array}{ccc|c} -1 & 1 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right)$$

$$\Longleftrightarrow \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

x_3 is a free variable, $x_1 = x_3$, $x_2 = 0$.

$$\vec{x} = \begin{pmatrix} x_3 \\ 0 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

The eigenvectors of A with eigenvalue $\lambda = 3$ are the vectors

$$\vec{x} = a \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \quad \text{where } a \neq 0.$$

$\boxed{\lambda = 2}$ We solve $(A - 2I)\vec{x} = \vec{0}$

$$\Longleftrightarrow \left(\begin{array}{ccc|c} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right)$$

$$\Longleftrightarrow \left(\begin{array}{ccc|c} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

x_1, x_3 are free variables $\quad x_2 = -x_3$.

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ -x_3 \\ x_3 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

The eigenvectors of A with eigenvalue $\lambda = 2$ are the vectors

$$\vec{x} = a \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + b \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

where a, b are not both zero.