Bonus Problem (Due 02-14-14) Solution

① If $ad-bc \neq 0$ then the system

$$(*)\begin{cases} ax+by=e \\ cx+dy=f \end{cases}$$

has the unique solution

$$x = \frac{de-bf}{ad-bc} = \frac{\begin{vmatrix} e & b \\ f & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}, \quad y = \frac{af-ce}{ad-bc} = \frac{\begin{vmatrix} a & e \\ c & f \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}.$$

PROOF: Suppose x, y satisfies $(*)$. Then

$$adx + bdy = de$$
$$bcx + bdy = bf.$$

Subtracting:

$$(ad-bc)x = de-bf, \quad x = \frac{de-bf}{ad-bc} = \frac{\begin{vmatrix} e & b \\ f & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}.$$

Similarly

$$acx + bcy = ce$$
$$acx + ady = af.$$

Subtracting:

$$(ad-bc)y = af-ce, \quad y = \frac{af-ce}{ad-bc} = \frac{\begin{vmatrix} a & e \\ c & f \end{vmatrix}}{\begin{vmatrix} a & d \\ c & d \end{vmatrix}}.$$

Conversely, if

$x = \dfrac{de-bf}{ad-bc}$, $y = \dfrac{af-ce}{ad-bc}$ we have

$$ax + by = \frac{a(de-bf) + b(af-ce)}{ad-bc}$$

$$= \frac{ade - bce}{ad-bc} = \frac{e(ad-bc)}{(ad-bc)} = e, \text{ and}$$

$$cx + dy = \frac{c(de-bf) + d(af-ce)}{ad-bc}$$

$$= \frac{adf - bcf}{ad-bc} = f\frac{(ad-bc)}{(ad-bc)} = f.$$

Hence (*) has the unique solution

$$x = \frac{de-bf}{ad-bc}, \quad y = \frac{af-ce}{ad-bc}. \quad \square$$

(2) If $ad-bc = 0$ then the system

$$(*) \begin{cases} ax + by = 0 \\ cx + dy = 0 \end{cases}$$

has infinitely many solutions

PROOF Suppose $ad-bc=0$; ie. $ad=bc$

Let $S_1 = \{(x,y) = (-bt, at): t \in \mathbb{R}\}$,

$S_2 = \{(x,y) = (-dt, ct): t \in \mathbb{R}\}$.

We show every element of S_1 is a solution of (*)
Suppose $(x,y) \in S_1$ then $x = -bt$, $y = at$ some $t \in \mathbb{R}$.
So $ax + by = -abt + abt = 0$,
$cx + dy = -bct + adt = -bct + bct = 0$.

Hence every element of S_1 is a solution of (*).
Similarly every element of S_2 is a solution of (*) (EX).

Case1 $a \neq 0$ or $b \neq 0$. Then S_1 has infinitely many elements. So (*) has infinitely many solutions

Case2 $c \neq 0$ or $d \neq 0$. Then S_2 has infinitely many elements. So (*) has infinitely many solutions.

Case3 $a = b = c = d = 0$.
Clearly all (x, y) is a solution of (*) & (*) has infinitely many solutions.

In all cases (*) has infinitely many solutions.