Solution to EXAM 1

(1) [11 points]

Complete the following:

(a) <u>The Existence and Uniqueness Theorem for 1st Order IVPs</u>

Consider the IVP

$$\frac{dy}{dx} = f(x,y), \qquad y(x_0) = y_0.$$

If $f(x,y)$ and $\partial f/\partial y$ are continuous on some open rectangle

$R = \{(x,y) : a < x < b,\ c < y < d\}$

that contains the point (x_0, y_0), then the IVP has a unique solution $\phi(x)$ on some interval $x_0 - \delta < x < x_0 + \delta$ where $\delta > 0$.

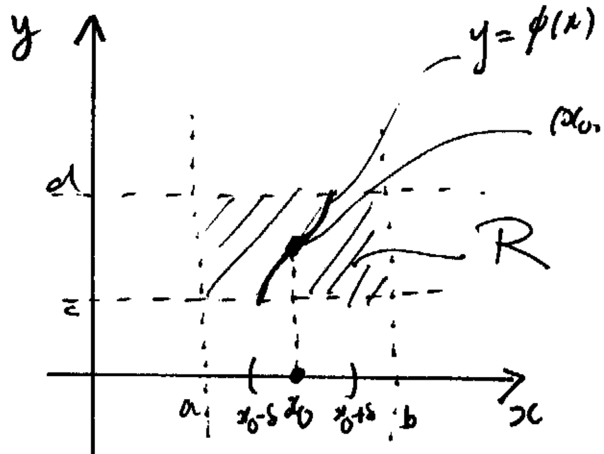


Diagram should include the point (x_0, y_0) and a generic solution $y = \phi(x)$

(b) The general solution of the <u>seperable equation</u>

$$f(y)\frac{dy}{dx} = g(x)$$

is given implicitly by

$$F(y) = G(x) + c,$$

where c is any constant,

$F(y) = \int f(y)\,dy$, and

$G(x) = \int g(x)\,dx$.

(c) To solve the linear equation

$$\frac{dy}{dx} + p(x)\, y = q(x), \qquad (*)$$

(i) We first let $\mu(x) = e^{P(x)}$,
where $P(x) = \int p(x)\,dx$.

(ii) Then we multiply both sides of (*) by $\mu(x)$
and write the resulting equation as

$$\frac{d}{dx}\left(\mu(x)\, y \right) = \mu(x)\, q(x).$$

(iii) The general solution of (*) is given by

$$y = \frac{1}{\mu(x)}\left(\int \mu(x) q(x)\,dx + C \right),$$

where C is any constant.

(d)
(i) A first order DE is exact if it can be written in the form

$$(*) \qquad \frac{d}{dx}\left(F(x,y) \right) = 0,$$

where $y = y(x)$.

(ii) (*) is equivalent to the DE

$$(**) \qquad M(x,y) + N(x,y)\frac{dy}{dx} = 0,$$

where

$$M(x,y) = \frac{\partial F}{\partial x},$$

and

$$N(x,y) = \frac{\partial F}{\partial y}.$$

(iii) The general solution of (*) is given implicitly by

$$F(x, y) = c,$$

where c is any constant.

(iv) If the first order partial derivatives of of $M(x,y)$ and $N(x,y)$ are continuous and (**) is exact then

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

(e) <u>Theorem</u> If the first order partial derivatives of of $M(x,y)$ and $N(x,y)$ are continuous on an open rectangle R, and

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

for all (x,y) in R, then the DE

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0$$

is exact on R.

(f) <u>Bernoulli's Equation</u> has the form

$$(*) \quad \frac{dy}{dx} + P(x)y = y^n Q(x),$$

where $P(x)$, $Q(x)$ are continuous functions on an interval (a,b).
To solve (*) we use the substitution

$$v = y^{1-n}, \quad \text{so that}$$

$$\frac{dv}{dx} = (1-n)y^{-n}\frac{dy}{dx}.$$

Then we multiply both sides of (*) by y^{-n}

(p.5)

to obtain the DE for $v = v(x)$: $y^{-n}\frac{dy}{dx} + P(x)y^{1-n} = Q(x)$

$$\frac{1}{(1-n)}\frac{dv}{dx} + P(x)\, v = Q(x)$$

which is linear.

(g) A homogeneous equation has the form

$$(*) \quad \frac{dy}{dx} = f\left(y/x\right).$$

To solve $(*)$ we let

$v = y/x$, so that $y = xv$,

and $\frac{dy}{dx} = x\frac{dv}{dx} + v$. In this way we can write $(*)$ as $x\frac{dv}{dx} = f(v) - v$, which is a seperable equation.

(2) [3 points]

Suppose the equation $y^2 + \ln y = \sin(x) + 1$ defines y implicitly as a function of x. By using implicit differentiation (or otherwise) find an ordinary first order DE satisfied by y. Then

$$\frac{d}{dx}\left(y^2 + \ln y\right) = \frac{d}{dx}\left(\sin(x) + 1\right),$$

$$2y\frac{dy}{dx} + \frac{1}{y}\frac{dy}{dx} = \cos x, \quad \&$$

$$\left(2y + 1/y\right)\frac{dy}{dx} = \cos x$$ is the required DE.

(3) [5 points]

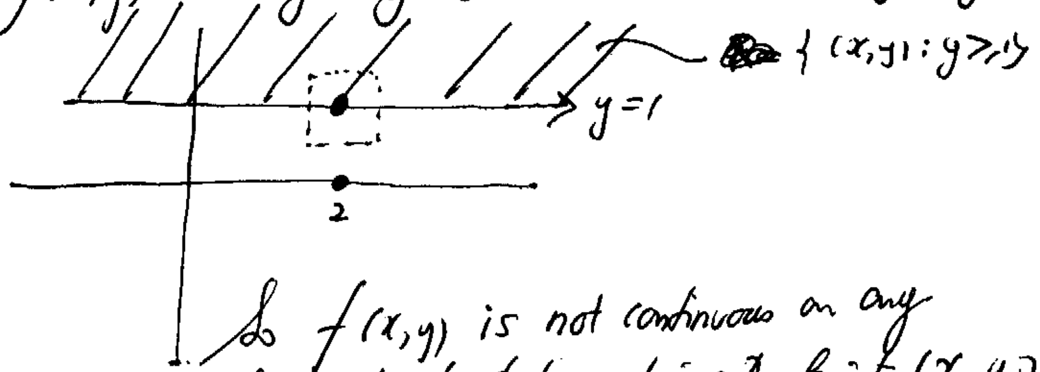
Consider the IVP

$$\frac{dy}{dx} = x^2 y + (y-1)^{3/2}, \qquad y(2) = 1.$$

Discuss the statement

" $f(x,y) = x^2y + (y-1)^{3/2}$ and $\frac{\partial f}{\partial y}$ are both continuous at $(2,1)$ so the IVP must have a unique solution $\phi(x)$ on some interval containing $x=2$, according to the Theorem in Question (1)(a). "

$f(x,y) = x^2y + (y-1)^{3/2}$ is continuous for $y \geq 1$.



So $f(x,y)$ is not continuous on any open rectangle that contains the point $(x_0,y_0) = (2,1)$.

Hence the Theorem does not apply & no conclusion can be drawn.

The statement is false.

<u>Note:</u> To be able to apply the Theorem f & $\frac{\partial f}{\partial y}$ must both be continuous on an <u>open</u> rectangle that contains (x_0, y_0).

(4) [6 points]

Solve the IVP

$$x\frac{dy}{dx} + 4y = 1 - 7x^3, \qquad y(1) = \frac{1}{4}.$$

$$\Leftrightarrow \quad \frac{dy}{dx} + \frac{4}{x}y = \frac{1}{x}(1-7x^3) \quad \text{assuming } x>0.$$

$$\int \frac{4}{x}\,dx = 4\ln x \;\;\&\;\; \mu(x) = e^{4\ln x} = (e^{\ln x})^4 = x^4.$$

We multiply both sides by $\mu(x)$:

$$x^4\frac{dy}{dx} + 4x^3 y = x^3(1-7x^3)$$

$$\Leftrightarrow \quad \frac{d}{dx}(x^4 y) = x^3 - 7x^6$$

$$x^4 y = \int (x^3 - 7x^6)\,dx$$

$$x^4 y = \frac{x^4}{4} - x^7 + c.$$

$y(1) = 1/4$. So $\frac{1}{4} = \frac{1}{4} - 1 + c$ & $c = 1$.

Hence $x^4 y = \frac{x^4}{4} - x^7 + 1$ &

the solution is given explicitly by

$$y = \frac{1}{x^4}\left(\frac{x^4}{4} - x^7 + 1\right),$$

for $x > 0$.

(5) [10 points]

Solve the IVP

$$\underbrace{(e^x y + x e^x y)}_{M} + \underbrace{(xe^x+2)}_{N}\frac{dy}{dx} = 0, \qquad y(0) = -1.$$

$$\frac{\partial m}{\partial y} = e^x + x e^x = \frac{\partial N}{\partial x} \quad \text{for all } (x,y).$$

This implies the equation is exact since the first order partial derivatives of m, N are continuous everywhere as well.

So there is a function $F(x,y)$ such that

$$\frac{\partial F}{\partial x} = M = e^x y + x e^x y \quad \& \quad \frac{\partial F}{\partial y} = N = xe^x + 2.$$

$$F = \int (xe^x + 2)\, dy = y(xe^x+2) + k(x).$$

$$\frac{\partial F}{\partial x} = y(xe^x + x) + k'(x) = e^x y + xe^x y,$$

$$k'(x) = 0.$$

We take $k(x) = 0$ & $F(x,y) = y(xe^x+2)$.

The general soln is given by

$$y(xe^x+2) = c,$$

where c is constant. $y(0) = -1$ & $-2 = c$.

Thus the soln is given explicitly by

$$y = \frac{-2}{(xe^x+2)}.$$

[5 points]
(6) Find the general solution of the DE

$$\frac{dy}{dx} = \frac{\sec^2 y}{1+x^2}.$$

$$\int \frac{dy}{\sec^2 y} = \int \frac{dx}{1+x^2}$$

$$\int \cos^2 y \, dy = \int \frac{dx}{1+x^2}$$

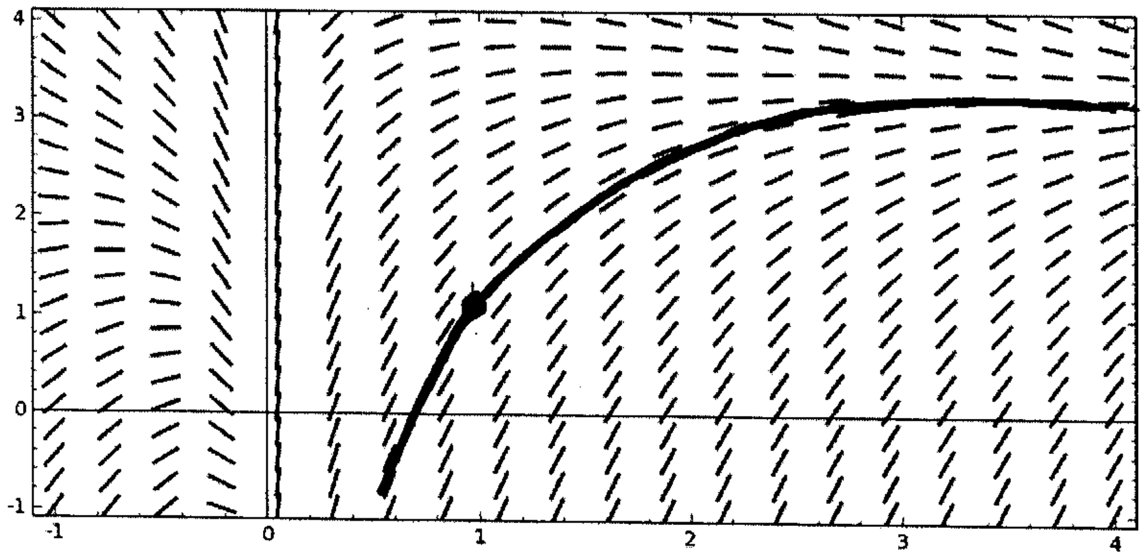
$$\Leftrightarrow \quad \frac{1}{2}\cos y \sin y + \frac{y}{2} = \tan^{-1} x + c'$$ (from Table of Integrals)

General Soln given implicitly by

$$\cos y \sin y + y = 2\tan^{-1} x + c,$$

where c is any constant.

(7) [3+1=4 points]
Below is the direction field plot of $\frac{dy}{dx} = 3 - y + \frac{1}{x}$.



On this diagram plot the solution that satisfies $y(1)=1$.
What do you notice about $\lim_{x\to\infty} y(x)$?

It seems that $\lim_{x\to\infty} y(x) = 3$.

(8) [6 points]
Find the general solution of the DE

$$(*) \qquad \frac{dy}{dx} = \frac{xy+y^2}{x^2}. \qquad \text{(Homogeneous)}$$

$$= \frac{y}{x} + \left(\frac{y}{x}\right)^2.$$

Let $v = y/x$ so $y = xv$ & $\frac{dy}{dx} = x\frac{dv}{dx} + v$

$(*) \Leftrightarrow \quad x\frac{dv}{dx} + v = v + v^2$ &

$$x\frac{dv}{dx} = v^2 \qquad \text{(seperable)}.$$

Q.1(I)

$$\Leftrightarrow \int \frac{1}{v^2}\,dv = \int \frac{1}{x}\,dx$$

$$-v^{-1} = \ln|x| + c'$$

$$v^{-1} = c - \ln|x|$$

$$\frac{y}{x} = v = \frac{1}{c-\ln|x|}$$

Soln given explicitly by

$$y = \frac{x}{c-\ln|x|},$$

where c is any constant.