Solution to MAP 2302 Exam 2 (Spring 2014)

(1.)

Complete the following:

1 (a) Definition: Two functions $y_1(t)$, $y_2(t)$ (defined on an interval I) are linearly dependent on I if there are constants c_1, c_2 not both zero such that

$$c_1 y_1(t) + c_2 y_2(t) = 0$$

for all $t \in I$. Otherwise, they are linearly independent on I.

1 (b) Lemma Two functions $y_1(t)$, $y_2(t)$ (on I) are linearly dependent on I if and only if one function is a constant multiple of the other.

1 (c) Theorem Suppose $y_1(t)$, $y_2(t)$ are differentiable and linearly dependent functions on an open interval I. Then

$$W[y_1, y_2](t) = 0,$$

for all $t \in I$.

1 (d) Corollary Suppose $y_1(t)$, $y_2(t)$ are differentiable functions on an open interval I and

$$W[y_1, y_2](t_0) \neq 0,$$

for some t_0 in I. Then $y_1(t)$, $y_2(t)$ are linearly independent on I.

2 (e) General Existence & Uniqueness Theorem for 2nd Order Linear DEs Let $a(t)$, $b(t)$, $c(t)$, $f(t)$ be continuous functions on an open interval I and suppose $a(t) \neq 0$ for all t in I. Suppose $t_0 \in I$ and Y_0, Y_1 are constants. Then

the IVP

$a(t)y'' + b(t)y' + c(t)y = f(t)$, $y(t_0) = Y_0$, $y'(t_0) = Y_1$,

has a unique solution valid on the interval I.

2 (f) Theorem Let $a(t)$, $b(t)$, $c(t)$ be continuous functions on an open interval I and $a(t) \neq 0$ for all t in I. Suppose $t_0 \in I$ and Y_0, Y_1 are constants. Suppose $y_1(t)$, $y_2(t)$ are solutions of the DE

(*) $a(t)y'' + b(t)y' + c(t)y = 0$.

(i) Suppose

$$W[y_1, y_2](t_0) = 0.$$

Then $y_1(t)$, $y_2(t)$ are linearly dependent on I and hence

$$W[y_1, y_2](t) = 0$$

for all t in I.

(ii) Suppose $y_1(t)$, $y_2(t)$ are linearly independent on I. Then

$$W[y_1, y_2](t) \neq 0$$

for all t in I.

(iii) The general solution of (*) is given by

$$y = c_1 y_1(t) + c_2 y_2(t)$$

where c_1, c_2 are any constants.

1 (g) A 2^{nd} order linear differential operator has the form

$$L[y] = y'' + p_1(t)y' + p_2(t)y,$$

where $p_1(t)$, $p_2(t)$ are given continuous functions on an interval I. If $y_1(t)$, $y_2(t)$ are twice differentiable functions & c_1, c_2 are constants, then

$$L[c_1 y_1 + c_2 y_2] = c_1 L[y_1] + c_2 L[y_2].$$

2 (h) Theorem on Nonhomogeneous 2^{nd} Order Linear DEs

Let $a(t), b(t), c(t), f(t)$ be continuous functions on an open interval I and suppose $a(t) \neq 0$ for all $t \in I$. Suppose $y_1(t), y_2(t)$ are two linearly independent solutions of

$$(*) \quad a(t)y'' + b(t)y' + c(t)y = 0,$$

and $y_p(t)$ is a particular solution of

$$(**) \quad a(t)y'' + b(t)y' + c(t)y = f(t).$$

Then the general solution of (**) is given by

$$y = y_p(t) + c_1 y_1(t) + c_2 y_2(t),$$

where c_1, c_2 are any constants. Further, the IVP

$$(***) \quad a(t)y'' + b(t)y' + c(t)y = f(t), \quad y(t_0) = Y_0, \quad y'(t_0) = Y_1,$$

has a unique solution for given constants Y_0, Y_0, and t_0 in I.

1 (i) The Superposition Principle

Suppose $y_1(t)$ is a solution to

(1) $a(t)y'' + b(t)y' + c(t)y = g_1(t)$,

and $y_2(t)$ is a solution to

(2) $a(t)y'' + b(t)y' + c(t)y = g_2(t)$

Then

$$y = c_1 y_1(t) + c_2 y_2(t), \quad \text{where } c_1, c_2 \text{ are constants},$$

is a solution to

(3) $a(t)y'' + b(t)y' + c(t)y = c_1 g_1(t) + c_2 g_2(t)$.

(2)

Solve the IVP

$$y'' + y' - 2y = 0, \qquad y(0) = 5, \; y'(0) = -1.$$

A.E: $r^2 + r - 2 = 0$

$(r+2)(r-1) = 0$

$r = 1, -2.$

$y = c_1 e^t + c_2 e^{-2t}$ for c_1, c_2 constants.

$y(0) = c_1 + c_2 = 5$

$y' = c_1 e^t - 2c_2 e^{-2t}$

$y'(0) = c_1 - 2c_2 = -1$

$c_1 + c_2 = 5$

$c_1 - 2c_2 = -1$

$3c_2 = 6$, $c_2 = 2$, $c_1 = 3$.

So the soln to the IVP is

$$y = 3e^t + 2e^{-2t}.$$

(3)

Use the method of variation of parameters to find a particular solution of

$$y'' + y = \sec t.$$

[HINT: $v_1' y_1 + v_2' y_2 = 0$ & $v_1' y_1' + v_2' y_2' = g(t)/a(t)$]

Ae: $r^2+1=0$, $r^2=-1$, $r=\pm i$. So

$y_1 = \cos t$, $y_2 = \sin t$ are lin. indept. sols. of the homog. DE.

We let

$$y_p = v_1 y_1 + v_2 y_2 \quad \text{and solve}$$

$$\begin{cases} v_1' y_1 + v_2' y_2 = 0 \\ v_1' y_1' + v_2' y_2' = \sec t \end{cases} \Leftrightarrow \begin{cases} \cos t\, v_1' + \sin t\, v_2' = 0 \\ -\sin t\, v_1' + \cos t\, v_2' = \sec t \end{cases}$$

We mult. eqn. 1 by $\sin t$ & eqn. 2 by $\cos t$:

$$\begin{cases} \sin t \cos t\, v_1' + \sin^2 t\, v_2' = 0 \\ -\sin t \cos t\, v_1' + \cos^2 t\, v_2' = 1 \end{cases}$$

Adding:

$$v_2' = 1, \quad v_2 = t.$$

$$\cos t\, v_1' = -\sin t\, v_2'$$

$$v_1' = \frac{-\sin t}{\cos t}, \quad v_1 = \int \frac{-\sin t}{\cos t}\,dt = \ln|\cos t|$$

(since $\frac{d}{dt}\cos t = -\sin t$).

Thus a particular soln is

$$y_p = \ln|\cos t| \cos t + t \sin t.$$

(4)
Determine the form of a particular solution for the differential equation

$$(*) \quad y'' - 4y' + 5y = e^{5t} + t\sin 3t - \cos 3t.$$

Do not solve the DE but do explain your reasoning.

A.E.: $r^2 - \cancel{4r} + 5 = 0$

$r^2 - 4r = -5$, $r^2 - 4r + 4 = -1$

$(r-2)^2 = -1$, $r - 2 = \pm i$, $r = 2 \pm i$ are roots of A.E.

A particular soln of

(A) $y'' - 4y' + 5y = e^{5t}$

has the form $y_p = Ae^{5t}$ since $r = 5$ is not a root of the auxiliary eqn.

A particular soln of

(B) $y'' - 4y' + 5y = t\,\cancel{\cos}\sin 3t - \cos 3t$

has the form

$$y_p = (Bt + C)\cos 3t + (Dt + E)\sin 3t$$

since $r = 3i$ is not a root of the auxiliary eqn & t^1 is largest power of t.

By Superposition Principle the form of a particular soln of $(*)$ is

$$y_p = Ae^{5t} + (Bt + C)\cos 3t + (Dt + E)\sin 3t$$

for some constants A, B, C, D, E.

(5)

You are given that
$$f_1(t) = \tfrac{1}{2}t - \tfrac{1}{2}$$
is a solution to
$$y'' + 2y' + 2y = t, \quad \text{and}$$
$$f_2(t) = -\tfrac{2}{5}\cos t + \tfrac{1}{5}\sin t$$
is a solution to
$$y'' + 2y' + 2y = \sin t.$$
Find the general solution of
$$y'' + 2y' + 2y = 2t - 5\sin t.$$

Let $L[y] = y'' + 2y' + 2y$. Then
$L[f_1] = t$ & $L[f_2] = \sin t$. So
$L[2f_1 - 5f_2] = 2L[f_1] - 5L[f_2] = 2t - 5\sin t$ &
$$y_p = 2f_1 - 5f_2 = 2\left(\tfrac{1}{2}t - \tfrac{1}{2}\right) - 5\left(-\tfrac{2}{5}\cos t + \tfrac{1}{5}\sin t\right)$$
$$= (t-1) + 2\cos t - \sin t$$
is a particular soln.

AE: $r^2 + 2r + 2 = 0$
$$r^2 + 2r + 1 = -2 + 1$$
$(r+1)^2 = -1$, $r + 1 = \pm i$, $r = -1 \pm i$, &
$y_1 = e^{-t}\cos t$, $y_2 = e^{-t}\sin t$ are lin. indept. solns of the homog. eqn.
Hence the general soln is given by
$$y = y_p + c_1 y_1 + c_2 y_2$$
$$= (t-1) + 2\cos t - \sin t + e^{-t}(c_1\cos t + c_2 \sin t),$$
where c_1, c_2 are any constants.

(6)

Discuss what the Existence and Uniqueness Theorem for Second Order Linear DEs implies about the solutions to the IVP

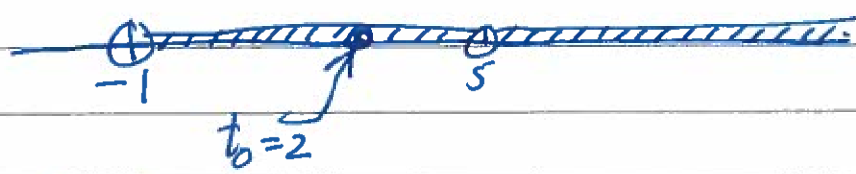
$$(t-5)y'' - 4y' + ty = \ln(1+t), \quad y(2)=2,\ y'(2)=3.$$

Explain your reasoning.

$$\Rightarrow \quad y'' - \frac{4}{(t-5)}y' + \frac{t}{t-5}y = \frac{\ln(1+t)}{t-5} \qquad (\text{for } t\neq 5).$$

$p(t) = \frac{-4}{t-5}$ & $q(t) = \frac{t}{t-5}$ are continuous for $t\neq 5$.

$g(t) = \frac{\ln(1+t)}{t-5}$ is continuous for $t>-1$ provided $t\neq 5$



All three functions $p(t), q(t), g(t)$ are continuous on the open interval $(-1,5)$ which contains $t_0=2$ (this is the largest such open interval).

The theorem implies that the IVP has a unique solution valid on the interval $(-1,5)$ ie $-1<t<5$.

(7)

Suppose $p(t)$, $q(t)$ are continuous functions for $-1 < t < 1$. Determine whether the following functions can be Wronskians on $(-1,1)$ for a pair of solutions $y_1(t), y_2(t)$ to the DE

$$y'' + p(t)y' + q(t)y = 0.$$

Explain your reasoning.

(a) $w(t) = 2t+1$ (b) $w(t) = t$

(c) $w(t) = \frac{1}{t^2+1}$ (d) $w(t) = 0$

$W[y_1,y_2](t)$ is either never zero on $(-1,1)$ in which case y_1, y_2 are lin. indep. solns; or $W[y_1,y_2](t) = 0$ for all $-1<t<1$ in which case y_1, y_2 are linearly dependent solutions.

Thus (a), (b) could not be wronskians since in (a) $W(-\frac{1}{2}) = 0$ & $-\frac{1}{2} \in (-1,1)$ & $w \not\equiv 0$; and in (b) $w(t) = 0$ if $t = 0$ (& $w \not\equiv 0$).

(c) could be a wronskian as it is never zero.

(d) is a wronskian if y_1, y_2 are linearly dependent solns.