(1) [ 1 + 2 + 1 + 1 + 2 + 1 + 2 = 10 points]

Complete the following.

1 (a) Let $f(t)$ be a function on $[0,\infty)$. The Laplace transform of $f$ is the function $F$ defined by the integral

$$F(s) := \int_0^\infty e^{-st} f(t)\,dt.$$

The domain of $F$ is the set of values of $s$ for which the integral converges. The Laplace transform of $f$ is denoted by both $F$ and $\mathcal{L}\{f\}$.

2 (b) A function $f(t)$ is said to be piecewise continuous on a finite interval $[a,b]$ if $f(t)$ is continuous at every point in $[a,b]$ except for a finite number of points at which $f(t)$ has a jump discontinuity. A function $f(t)$ is piecewise continuous on $[0,\infty)$ if $f(t)$ is piecewise continuous on $[0,b]$ for each $b>0$.

1 (c) A function $f(t)$ is said to be of exponential order $\alpha$ if there are positive constants $T$ and $M$ such that $|f(t)| \le Me^{\alpha t}$ for $t \ge T$.

1 (d) Theorem If $f(t)$ is piecewise continuous on $[0,\infty)$ and of exponential order $\alpha$, then $\mathcal{L}\{f\}(s)$ exists for $s > \alpha$.

2 (e) A function $f$ is analytic at $x_0$ if $f(x)$ is equal to a power series

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n \quad \text{for } |x - x_0| < R$$

for some $R > 0$, and the power series has positive radius of convergence $R$ i.e. the power series converges for $|x - x_0| < R$.

1 (f) a point $x_0$ is called an ordinary point of

$$y'' + p(x)y' + q(x)y = 0$$

if both $p(x)$ and $q(x)$ are analytic at $x_0$.
If $x_0$ is not an ordinary point it is called a singular point.

2 (g) Theorem Suppose $x_0$ is an ordinary point of
(*) $y'' + p(x)y' + q(x)y = 0$.

Then (*) has two linearly independent analytic solutions of the form

(**) $y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$.

Moreover the radius of convergence of any power series solution given by (**) is at least the distance from $x_0$ to the nearest singularity real or complex.

(2)

(a) $\mathcal{L}\{t\sin 2t\} = \dfrac{4s}{(s^2+4)^2}$ (# 14)

$\mathcal{L}\{te^{-t}\sin 2t\} = \dfrac{4(s+1)}{((s+1)^2+4)^2}$ (# 7).

(b) $\cos(a+b) = \cos a\cos b - \sin a\sin b$

$\cos(a-b) = \cos a\cos b + \sin a\sin b$

$\sin a\sin b = \frac{1}{2}\left(\cos(a-b) - \cos(a+b)\right)$

$\sin 2t\sin 3t = \frac{1}{2}(\cos t - \cos 5t)$

$\mathcal{L}\{\sin 2t\sin 3t\} = \frac{1}{2}\left(\dfrac{s}{s^2+1} - \dfrac{s}{s^2+25}\right)$ (# 6).

(3) Let $Y = \mathcal{L}(y)$.

Then $\mathcal{L}\{y'\} = sY - y(0) = sY - 1$.

$\mathcal{L}\{y''\} = s^2Y - sy(0) - y'(0) = s^2Y - s - 3$

$\mathcal{L}\{y'' - 2y' + y\} = \mathcal{L}\{\cos t - \sin t\}$

$\mathcal{L}\{y''\} - 2\mathcal{L}\{y'\} + \mathcal{L}\{y\} = \dfrac{s}{s^2+1} - \dfrac{1}{s^2+1}$

$(s^2Y - s - 3) - 2(sY - 1) + Y = \dfrac{s-1}{s^2+1}$

$(s^2 - 2s + 1)Y - s - 1 = \dfrac{s-1}{s^2+1}$

$$Y = \frac{1}{(s^2-2s+1)}\left( s+1 + \frac{(s-1)}{s^2+1} \right)$$

(4) (i) $g(t) = t\,(u(t-1) - u(t-2)) + u(t-2)$

$$= t\,u(t-1) + (1-t)u(t-2).$$

(ii) $\mathcal{L}\{g\} = \mathcal{L}\{t\,u(t-1)\} + \mathcal{L}\{(1-t)u(t-2)\}$

$$= e^{-s}\mathcal{L}\{t+1\} + e^{-2s}\mathcal{L}\{(1-(t+2))\}$$

$$= e^{-s}\left(\frac{1}{s} + \frac{1}{s^2}\right) + e^{-2s}\mathcal{L}\{-1-t\}$$

$$= e^{-s}\left(\frac{1}{s} + \frac{1}{s^2}\right) + e^{-2s}\left(\frac{-1}{s} - \frac{1}{s^2}\right)$$

$$= \left(\frac{1}{s} + \frac{1}{s^2}\right)\left(e^{-s} - e^{-2s}\right).$$

(5)

$$\frac{3+2s}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2}$$

$$3+2s = A(s+2) + B(s+1)$$

$s=-1$: $1 = A$

$s=-2$ $3-4 = -B$, $B=1$

So $\mathcal{L}^{-1}\left\{\frac{3+2s}{(s+1)(s+2)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} = e^{-t} + e^{-2t}$

$$\frac{1}{(s+1)(s+2)(s+3)} = \frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{s+3}$$

$$1 = A(s+2)(s+3) + B(s+1)(s+3) + C(s+1)(s+2)$$

$s=-1$: $\quad 1 = A(1)(2), \quad A = 1/2$

$s=-2$ $\quad 1 = B(-1)(1) \quad B = -1$

$s=-3$ $\quad 1 = C(-2)(-1) \quad C = 1/2.$

$$f(t) = \mathcal{L}^{-1}\left\{\frac{e^{-6}}{(s+1)(s+2)(s+3)}\right\} = e^{-6}\left[\mathcal{L}^{-1}\left\{\frac{1}{2}\frac{1}{s+1}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{2}\frac{1}{s+3}\right\}\right]$$

$$= e^{-6}\left[\frac{1}{2}e^{-t} - e^{-2t} + \frac{1}{2}e^{-3t}\right]$$

We use $\mathcal{L}\{u(t-t_0)f(t-t_0)\} = e^{-st_0}\mathcal{L}\{f\}.$

$$\mathcal{L}^{-1}\left\{\frac{e^{-2s-6}}{(s+1)(s+2)(s+3)}\right\}$$

$$= \cancel{e^{+e6}}\, u(t-2)f(t-2)$$

$$= u(t-2)\left[\frac{1}{2}e^{-(t-2)} - e^{-2(t-2)} + \frac{1}{2}e^{-3(t-2)}\right]e^{-6}$$

$$= u(t-2)\left[\frac{1}{2}e^{-t-4} - e^{-2t-2} + \frac{1}{2}e^{-3t}\right]$$

Hence,

$$\mathcal{L}^{-1}\left\{\frac{3+2s}{(s+1)(s+2)} + \frac{e^{-2s}}{(s+1)(s+2)(s+3)}\right\}$$

$$= e^{-t} + e^{-2t} + u(t-2)\left(\tfrac{1}{2}e^{-t-4} - e^{-2t-2} + \tfrac{1}{2}e^{-3t}\right)$$

(6)

(a) Let $y = \sum_{n=0}^{\infty} a_n x^n$. Then

$$y' = \sum_{n=0}^{\infty} a_n \cdot n \cdot x^{n-1},$$

$$y'' = \sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}$$

$$= \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n,$$

$$-xy' = \sum_{n=0}^{\infty} n a_n x^n,$$

$$4y = \sum_{n=0}^{\infty} 4a_n x^n.$$

Hence,

$$y'' - xy' + 4y = \sum_{n=0}^{\infty}\left((n+2)(n+1)a_{n+2} - n a_n + 4a_n\right)x^n$$

$$= \sum_{n=0}^{\infty}\left((n+2)(n+1)a_{n+2} + (4-n)a_n\right)x^n$$

$$= 0$$

$$\Longleftrightarrow \quad (n+2)(n+1)a_{n+2} + (4-n)a_n = 0 \text{ for } n \geq 0$$

$$\Leftrightarrow \qquad a_{n+2} = \frac{(n-4)\,a_n}{(n+2)(n+1)} \quad \text{for } n \geq 0.$$

$$a_2 = \frac{-4}{2\cdot 1} a_0 = -2a_0,$$

$$a_3 = \frac{-3 \quad a_1}{3\cdot 2} = -\tfrac{1}{2} a_1$$

Thus

$$y = a_0 + a_1 x - 2a_0 x^2 - \tfrac{1}{2} a_1 x^3 + \cdots$$

where $a_0, a_1$ are any constants.

(b) $$a_4 = \frac{(-2) \quad a_2}{4\cdot 3} = \frac{(-4)(-2)}{4\cdot 3\cdot 2\cdot 1} a_0 = \frac{a_0}{3}$$

$$a_5 = \frac{(-1) \quad a_3}{5\cdot 4} = \frac{-1}{20}\left(\frac{-1}{2}\right) a_1$$

$$a_6 = \frac{0\cdot a_4}{6\cdot 5} = 0$$

$$a_7 = \frac{(1) \quad a_5}{7\cdot 6} = \frac{(-3)(-1)(1) \quad a_1}{7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2}$$

$$a_8 = \frac{2\,a_6}{8\cdot 7} = 0$$

$$a_9 = \frac{(3) \quad a_7}{9\cdot 8} = \frac{(-3)(-1)(1)(3)\,a_1}{9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot}$$

Thus $a_{2n} = 0$ for $n \geq 3$.

$$a_{2n+1} = \frac{(-3)(-1)(1)(3)\cdots(2n-5)}{(2n+1)!} a_1 \quad \text{for } n \geq 1.$$

(c)

$$y = \sum_{n=0}^{\infty} a_{2n} x^{2n} + \sum_{n=0}^{\infty} a_{2n+1} x^{2n+1}$$

$$= \left(a_0 - 2a_0 x^2 + \frac{a_0}{3} x^4\right)$$

$$\left( a_1 x^1 + \sum_{n=1}^{\infty} a_1 \frac{(-3)(-1)\cdots(2n-5)}{(2n+1)!} x^{2n+1} \right)$$

$$= a_0\left(1 - 2x^2 + \frac{1}{3} x^4\right)$$

$$+ a_1\left( x + \sum_{n=1}^{\infty} \frac{(-3)(-1)\cdots(2n-5)}{(2n+1)!} x^{2n+1}\right)$$

where $a_0, a_1$ are any constants.
Two linearly independent solutions are

$$y_1 = 1 - 2x^2 + \tfrac{1}{3} x^4,$$

$$y_2 = x + \sum_{n=1}^{\infty} \frac{(-3)(-1)\cdots(2n-5)}{(2n+1)!} x^{2n+1}.$$