Final Exam – Spring 2008

(8)(a) Let $Y = \mathcal{L}\{y\}$. Then

$$\mathcal{L}\{y''\} = s^2 Y - s y(0) - y'(0) = s^2 Y - 1.$$

$$\mathcal{L}\{y'' + y\} = \mathcal{L}\{u(t-3)\},$$

$$(s^2 Y - 1) + Y = \frac{e^{-3s}}{s}$$

$$(s^2+1) Y = 1 + \frac{e^{-3s}}{s}$$

$$Y = \frac{1}{s^2+1} + \frac{e^{-3s}}{s(s^2+1)}.$$

(b) $$\frac{1}{s(s^2+1)} = \frac{A}{s} + \frac{Bs+C}{s^2+1}.$$

$$1 = A(s^2+1) + (Bs+C)s.$$

$s=0$: $\quad 1 = A$

Coeff of s^2: $\quad 0 = A + B, \quad B = -1.$

Coeff of s: $\quad 0 = C$

So $$f(t) = \mathcal{L}^{-1}\left\{\frac{1}{s(s^2+1)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{s}{s^2+1}\right\}$$

$$= 1 - \cos t.$$

Thus $$\mathcal{L}^{-1}\left\{\frac{e^{-3s}}{s(s^2+1)}\right\} = u(t-3) f(t-3)$$

$$= u(t-3)\left(1 - \cos(t-3)\right).$$

Hence $$y = \mathcal{L}^{-1}\{Y\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} - \mathcal{L}^{-1}\left\{\frac{e^{-3s}}{s(s^2+1)}\right\}$$

$$= \sin t + u(t-3)\left(1 - \cos(t-3)\right).$$

(a) [THE "LONG" WAY].

Let $y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$

So $y' = a_1 + 2a_2 x + 3a_3 x^2 + \cdots$

$y'' = 2a_2 + 6a_3 x + \cdots$

$$
\begin{aligned}
2y'' &= 4a_2 + 12a_3 x + \cdots \\
x^2 y'' &= \qquad\qquad\qquad 2a_2 x^2 + \cdots \\
-xy' &= \qquad -a_1 x - 2a_2 x^2 + \cdots \\
-3y &= -3a_0 - 3a_1 x - 3a_2 x^2 + \cdots
\end{aligned}
$$

So

$(2+x^2)y'' - xy' - 3y$

$= (4a_2 - 3a_0) + (12a_3 - 4a_1)x + \cdots$

$= 0$

$\Leftrightarrow \quad 4a_2 = 3a_0, \quad 12a_3 = 4a_1, \ldots$

$\Leftrightarrow \quad a_2 = 3/4\, a_0, \quad a_3 = 1/3\, a_1, \ldots$

$y = a_0 + a_1 x + (3/4\, a_0)x^2 + \frac{a_1}{3} x^3 + \cdots$

$= a_0(1 + 3/4\, x^2 + \cdots)$

$+ a_1(x + \frac{1}{3}x^3 + \cdots)$

where a_0, a_1 are any constants.

NOTE: alternatively you could use (b) below to find a_2, a_3 etc.

(b) Let

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then
$$y' = \sum_{n=0}^{\infty} n a_n x^{n-1},$$

$$y'' = \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2}$$

$$= \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n.$$

$$2y'' = \sum_{n=0}^{\infty} 2(n+2)(n+1) a_{n+2} x^n;$$

$$x^2 y'' = \sum_{n=0}^{\infty} n(n-1) a_n x^n,$$

$$-xy' = \sum_{n=0}^{\infty} -n a_n x^n,$$

$$-3y = \sum_{n=0}^{\infty} -3 a_n x^n.$$

Thus,

$$(2+x^2)y'' - xy' - 3y$$

$$= \sum_{n=0}^{\infty} \left(2(n+2)(n+1) a_{n+2} + (n(n-1) - n - 3) a_n \right) x^n$$

$$= \sum_{n=0}^{\infty} \left(2(n+2)(n+1) a_{n+2} + (n^2 - 2n - 3) a_n \right) x^n$$

$$\Leftrightarrow \quad 2(n+2)(n+1) a_{n+2} = -(n^2-2n-3) a_n$$

$$\Leftrightarrow \quad 2(n+2)(n+1) a_{n+2} = -(n-3)(n+1) a_n,$$

for $n \geq 0$

$$\Leftrightarrow \quad a_{n+2} = -\frac{(n-3)(n+1)}{2(n+2)(n+1)} a_n$$

$$= -\frac{(n-3)}{2(n+2)} a_n, \quad \text{for } n \geq 0$$

(c)
$$a_2 = -\frac{(-3)}{2 \cdot 2} a_0$$

$$a_3 = -\frac{(-2)}{2 \cdot 3} a_1$$

$$a_4 = -\frac{(-1)}{2 \cdot 4} a_2 = \frac{(-3)(-1)}{2^2 \cdot 2 \cdot 4} a_0$$

$$a_5 = 0\, a_3 = 0$$

$$a_6 = -\frac{(1)}{2 \cdot 6} a_4 = -\frac{(-3)(-1)(1)}{2^3 \cdot 2 \cdot 4 \cdot 6} a_0$$

$$a_7 = -\frac{2}{2 \cdot 7} a_5 = 0$$

$$a_8 = -\frac{(3)}{2 \cdot 8} a_6 = \frac{(-3)(-1)(1)(3)}{2^4 \cdot 2 \cdot 4 \cdot 6 \cdot 8} a_0$$

Thus $a_{2n+1} = 0$ for $n \geq 2$ &

$$a_{2n} = (-1)^n \frac{(-3)(-1)(1)(3)\cdots(2n-5)}{2^n \cdot 2 \cdot 4 \cdot 6 \cdots (2n)} a_0$$

$$= \frac{(-1)^n (-3)(-1)\cdots\cdots(2n-5)}{2^{2n}\cdot n!} a_0$$

for $n \geq 1$.

Thus $$y = \sum_{n=0}^{\infty} a_{2n} x^{2n} + \sum_{n=0}^{\infty} a_{2n+1} x^{2n+1}$$

$$= a_0 + \sum_{n=1}^{\infty} \frac{(-1)^n (-3)(-1)\cdots(2n-5)}{2^{2n}\, n!} x^{2n} a_0$$

$$+ a_1 x + \tfrac{1}{3} a_1 x^3$$

$$= a_0 \left(1 + \sum_{n=1}^{\infty} \frac{(-1)^n (-3)(-1)\cdots(2n-5)\, x^{2n}}{2^{2n}\, n!} \right)$$

$$+ a_1 \left(x + \tfrac{1}{3} x^3 \right)$$ for any constant a_0, a_1.

So the linearly independent solutions are

$$y_1 = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n (-3)(-1)\cdots(2n-5) x^{2n}}{2^{2n}\, n!})$$

$$y_2 = x + \tfrac{1}{3} x^3.$$