MAP 2302 — FINAL EXAM — SPRING 2008

PERIOD: ________ **NAME:** ________________________

Instructions: All work should be written in a proper and coherent manner. Write in such a way that any student in the class can follow your work. When working problems show all your work. Answers with no work or explanations will receive no credit, unless otherwise specified. Tables of integrals and Laplace transforms are supplied. **ONLY SCIENTIFIC CALCULATORS ALLOWED**.

TOTAL POSSIBLE: $3 + 6 + 10 + 6 + 5 + 5 + 5 + 10 + 15 + 4 = 69$ points.

100% = 50 points

(1) [3 **points**] Suppose that $y = y(x)$ is given implicitly by the equation

$$e^{xy} + y^2 = x^2 - 1.$$

Find a *first order differential equation* satisfied by y.

(2) [6 **points**]

The Existence and Uniqueness Theorem for 1st Order DE's: Given the initial value problem

$$\frac{dy}{dx} = f(x, y), \qquad y(x_0) = y_0,$$

assume that f and $\partial f/\partial y$ are continuous functions on an open rectangle

$$R = \{(x, y) \,:\, a < x < b,\, c < y < d\}$$

that contains the point (x_0, y_0). Then the initial value problem has a unique solution $\phi(x)$ on some interval $x_0 - h < x < x_0 + h$, where h is some positive number.

Determine whether this Theorem implies that the initial value problem:

$$\frac{dy}{dx} = \frac{1}{x^3} + \frac{1}{y^3}, \qquad y(1) = 2,$$

has a unique solution on some open interval containing $x = 1$. Show all working and reasoning.

(3) [$5 + 5 = 10$ **points**] Solve the following initial value problems. If possible, give the solution explicitly.

(i) $$\frac{dy}{dx} = 2\sqrt{1 + y}\,\cos x, \qquad y(\pi) = 0,$$

(ii) $$x\frac{dy}{dx} = y + x^2 e^x, \qquad y(1) = e + 2.$$

(4) [6 **points**]

The Existence and Uniqueness Theorem for 3rd Order LINEAR DE's: Suppose that $p_2(x)$, $p_1(x)$, $p_0(x)$ and $q(x)$ are continuous functions on an interval (a,b) that contains the point x_0. Let γ_0, γ_1 and γ_2 be constants. Then the initial value problem

$$\begin{aligned} &y'''(x) + p_2(x)y''(x) + p_1(x)y'(x) + p_0(x)y(x) = q(x), \\ &y(x_0) = \gamma_0, \quad y'(x_0) = \gamma_1, \quad y''(x_0) = \gamma_2 \end{aligned}$$

has a unique solution $y(x)$ on the interval (a,b).

Determine the largest interval (a,b) for which this Theorem guarantees the existence of a unique solution on (a,b) to the initial value problem

$$\begin{aligned} &x\ln(1+x)\,y''' + \sqrt{1+x^2}\,y'' - xy' + y = 0, \\ &y(-1/2) = 0, \quad y'(-1/2) = 1, \quad y''(-1/2) = 2. \end{aligned}$$

Show all working and reasoning.

(5) [5 **points**] Verify that the functions $y_1 = x$, $y_2 = x^2$, $y_3 = x^3$ form a fundamental set of solutions for the differential equation

$$x^3y''' - 3x^2y'' + 6xy' - 6y = 0, \qquad x > 0,$$

and find the **general solution**.

(6) [5 **points**] Solve the initial value problem

$$y''(x) + y(x) = 2e^{-x}, \quad y(0) = 0, \quad y'(0) = 0.$$

(7) [$1+4=5$ **points**]

(a) Let $f(t)$ be a function defined on $[0,\infty)$. Define the **Laplace transform** of f.

(b) Determine the Laplace transform $\mathcal{L}\{e^t\}$ from first principles; ie. using only the definition of the Laplace transform.

(8) [$5+5=10$ **points**] For the following initial value problem

$$y'' + y = u(t-3), \quad y(0) = 0, \quad y'(0) = 1.$$

(a) find $Y(s)$ the Laplace transform of the solution $y(t)$.

(b) [BONUS] Find the solution $y(t)$.

(9) [$5+5+5=15$ **points**] Consider the differential equation

$$(2+x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - 3y = 0.$$

(a) Find at least the first four terms in the power series expansion about $x = 0$ of the general solution.

[HINT: First let $y = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots$.]

(b) [BONUS] Let

$$y = \sum_{n=0}^{\infty} a_n x^n$$

be a power series solution to differential equation. Show that

$$a_{n+2} = -\frac{(n-3)a_n}{2(n+2)} \quad \text{for } n \geq 0.$$

(c) [BONUS] Find two independent solutions explicitly.

[HINT: One solution is a polynomial.]

(10) [4 **points**] [BONUS]

Name each guy. For each person, describe some aspect of their life AND their mathematics. Write in complete sentences.

(A)  (B)