Homework 6

1.
$$\int e^{-st}\, t e^{t}\, dt = \int \underbrace{t}_{u} \underbrace{e^{-(s-1)t}\, dt}_{dv}$$

$$= u\,v - \int v\, du \quad \text{(integration by parts)}$$

$$= t\,\frac{e^{-(s-1)t}}{-(s-1)} - \int \frac{e^{-(s-1)t}}{-(s-1)}\, dt \quad (\text{assuming } s \neq 1)$$

$$= \frac{t e^{-(s-1)t}}{1-s} + \frac{1}{s-1} \int e^{-(s-1)t}\, dt$$

$$= \frac{t e^{-(s-1)t}}{1-s} + \frac{1}{s-1}\, \frac{e^{-(s-1)t}}{-(s-1)}$$

$$= \frac{t e^{-(s-1)t}}{1-s} - \frac{1}{(s-1)^2}\, e^{-(s-1)t}.$$

So
$$\int_0^N e^{-st}\, t e^{t}\, dt = \left. \frac{t e^{-(s-1)t}}{1-s} - \frac{1}{(s-1)^2}\, e^{-(s-1)t} \right]_{t=0}^{t=N}$$

$$= \frac{N e^{-(s-1)N}}{1-s} - \frac{1}{(s-1)}\, e^{-(s-1)N} - 0 + \frac{1}{(s-1)^2}$$

Assume $s > 1$ so that $-(s-1) < 0$.

Then $\lim_{N\to\infty} e^{-(s-1)N} = 0$ &

$$\lim_{N\to\infty} N e^{-(s-1)N} = \lim_{N\to\infty} \frac{N}{e^{(s-1)N}} \quad \left(\frac{\infty}{\infty}\right)$$

$$= \lim_{N\to\infty} \frac{1}{(s-1)\, e^{(s-1)N}} = 0 \quad \text{(by L'Hopital's Rule)}$$

Therefore (p.2)

$$\lim_{N\to\infty}\int_0^N e^{-st}\, t e^{t}\,dt = \lim_{N\to\infty}\left(\frac{Ne^{-(s-1)N}}{1-s} - \frac{1}{s-1}e^{-(s-1)N} + \frac{1}{(s-1)^2}\right)$$

$$= 0 - 0 + \frac{1}{(s-1)^2} \;\&$$

$$\mathcal{L}\{t e^{t}\} = \frac{1}{(s-1)^2} \text{ for } s>1.$$

(2)

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$$

$$= \int_0^1 e^{-st} f(t)\,dt + \int_1^2 e^{-st} f(t)\,dt + \int_1^\infty e^{-st} f(t)\,dt$$

$$= 0 + \int_1^2 e^{-st} e^{-t}\,dt + 0$$

$$= \int_1^2 e^{-(s+1)t}\,dt = \left.\frac{e^{-(s+1)t}}{(-(s+1))}\right]_{t=1}^{t=2} \quad \text{assuming } s>-1$$

$$= \frac{1}{(s+1)}\left(e^{-(s+1)} - e^{-2(s+1)}\right).$$

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(a) $f(t) = \sinh t = \dfrac{e^t - e^{-t}}{2}$

$|f(t)| = \frac{1}{2}|e^t - e^{-t}|$

$= \frac{1}{2} | e^t(1-e^{-2t}) | = \frac{1}{2}|e^t||1-e^{-2t}|$

$= \frac{1}{2} e^t (1-e^{-2t}) \leqslant \frac{1}{2} e^t$ for $t \geqslant 0$

since $0 \leqslant 1 - e^{-2t} < 1$.

Hence $f(t)$ has exponential order $\alpha = 1$.

(b) $g(t) = t^{\sqrt{t}} = e^{(\ln t)\sqrt{t}}$

Let $h(t) = \dfrac{(\ln t)\sqrt{t}}{t} = \dfrac{\ln t}{\sqrt{t}}$ for $t \geqslant 1$.

$h'(t) = \dfrac{\sqrt{t} \cdot 1/t - (\ln t)\frac{1}{2\sqrt{t}}}{t}$

$= \dfrac{2 - \ln t}{2t\sqrt{t}} = 0$ at $t = e^2$

h'		+	0	−
t	1		e^2	

So $h(t)$ has a maximum at $t = e^2$ &

$0 \leqslant h(t) \leqslant h(e^2) = \frac{2}{e} < 1$ for $t \geqslant 1$.

Therefore

$0 \leqslant \dfrac{(\ln t)\sqrt{t}}{t} < 1$ for $t \geqslant 1$ &

$0 \leqslant (\ln t)\sqrt{t} < t$ for $t \geqslant 1$.

Hence,
$$|g(t)| = e^{(\ln t)\sqrt{t}} < e^{t} \quad \text{for } t \geq 1 \text{ \&}$$

g has exponential order $\alpha = 1$.