TEST 3 – Spring 2008

2(a) The Laplace transform of $f(t)$ is the function F defined by
$$F(s) := \int_0^\infty e^{-st} f(t)\,dt.$$
The domain of F is the set of s for which the integral converges.

(b) $\mathcal{L}\{1\} = \int_0^\infty e^{-st}\,dt.$

$$\int e^{-st}\,dt = \frac{e^{-st}}{-s} \quad \text{assuming } s \neq 0.$$

So
$$\int_0^N e^{-st}\,dt = \left.\frac{e^{-st}}{-s}\right]_{t=0}^{t=N} = \frac{1}{s} - \frac{e^{-sN}}{s}$$

$$\lim_{N\to\infty} \int_0^N e^{-st}\,dt = \lim_{N\to\infty}\left(\frac{1}{s} - \frac{e^{-sN}}{s}\right)$$
$$= \frac{1}{s} - 0 \quad (\text{since if } s>0,\ -sN \to -\infty \text{ as } N\to\infty)$$
$$= 1/s,$$
if $s>0$.

Thus $\mathcal{L}\{1\} = \int_0^\infty e^{-st}\,dt = 1/s$ for $s>0$.

3.

(a) $\mathcal{L}\{3t^4 + e^{3t}\sin 2t - t\cos t\}$

$= 3\mathcal{L}\{t^4\} + \mathcal{L}\{e^{3t}\sin 2t\} - \mathcal{L}\{t\cos t\}$

$= \dfrac{3\cdot 4!}{s^5} + \dfrac{2}{(s-3)^2+4} - \dfrac{(s^2-1)}{(s^2+1)^2}$ for $s>3$.

$= \dfrac{72}{s^5} + \dfrac{2}{s^2-6s+13} - \dfrac{(s^2-1)}{(s^2+1)^2}$ for $s>3$.

(b) $\sin(a+b) = \sin a \cos b + \cos a \sin b,$

$\sin(a-b) = \sin a \cos b - \cos a \sin b,$

$\sin a \cos b = \frac{1}{2}\left(\sin(a+b) + \sin(a-b)\right).$

So

$$\mathcal{L}\{\sin 2t \cos 3t\} = \frac{1}{2}\mathcal{L}\{\sin 5t - \sin t\}$$

$$= \frac{1}{2}\left(\frac{5}{s^2+25} - \frac{1}{s^2+1}\right).$$

(c) $$\mathcal{L}\{t^2 \sin t\} = \frac{d^2}{ds^2}\mathcal{L}\{\sin t\} = \frac{d^2}{ds^2}\left(\frac{1}{s^2+1}\right)$$

$$= \frac{d}{ds}\frac{-2s}{(s^2+1)^2} = \frac{8s^2}{(s^2+1)^3} - \frac{2}{(s^2+1)^2}$$

for $s>0$.

(4) Let $Y = \mathcal{L}\{y\}$. Then

$\mathcal{L}\{y'\} = sY - y(0) = sY - 1,$

$\mathcal{L}\{y''\} = s^2Y - sy(0) - y'(0) = s^2Y - s + 2,$

$$\mathcal{L}\{y'' + 5y' + 6y\} = \mathcal{L}\{e^{2t}\cos t + \sin 2t\},$$

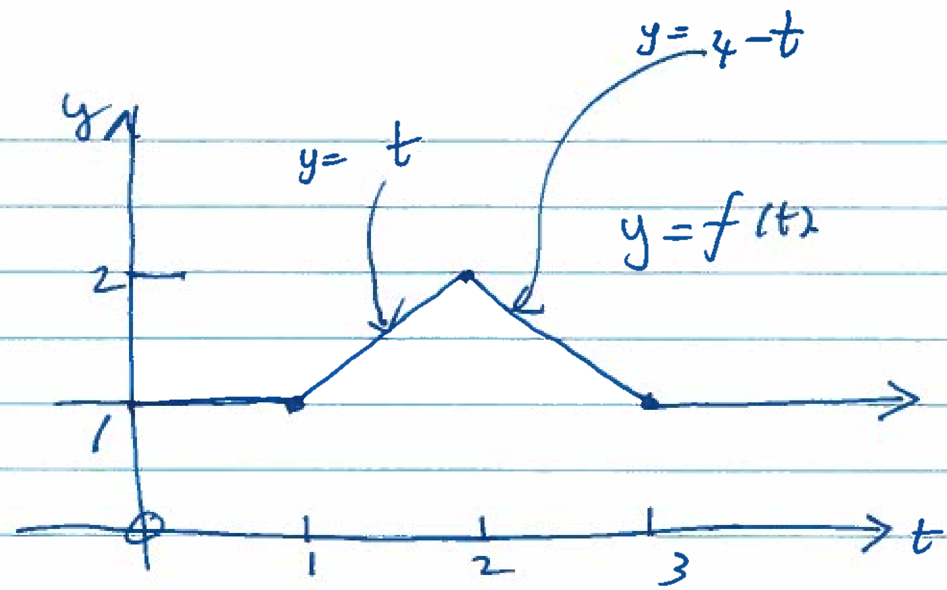
$$(s^2Y - s + 2) + 5(sY - 1) + 6Y = \frac{s-2}{(s-2)^2+1} + \frac{2}{s^2+4}$$

$$(s^2 + 5s + 6)Y = (s+3) + \frac{(s-2)}{(s-2)^2+1} + \frac{2}{s^2+4}$$

$$Y = \frac{1}{(s^2+5s+6)}\left[(s+3) + \frac{(s-2)}{(s-2)^2+1} + \frac{2}{s^2+4}\right].$$

(P.3)

(5)



$$f(t) = \begin{cases} 1 & \text{if } t < 1 \\ t & \text{if } 1 < t < 2 \\ 4-t & \text{if } 2 < t < 3 \\ 1 & \text{if } t > 3 \end{cases}$$

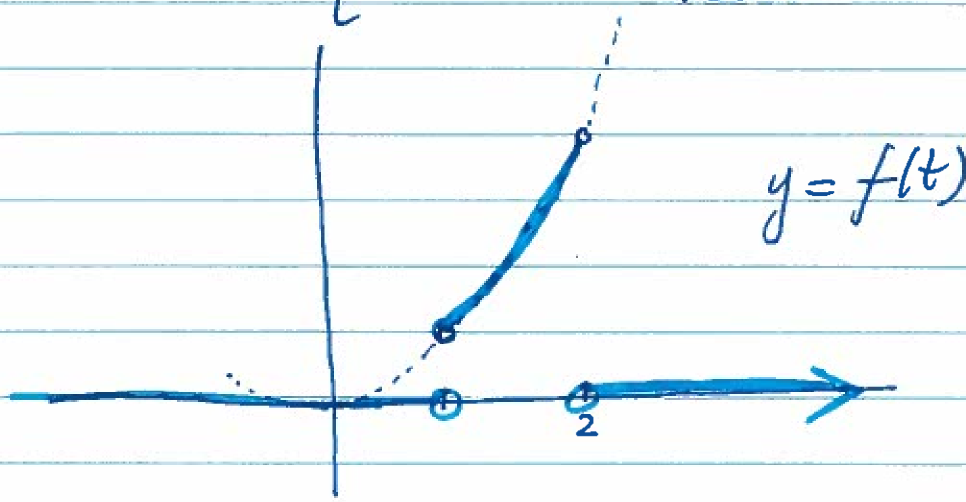
	$t<1$	$1<t<2$	$2<t<3$	$t>3$
$1-u(t-1)$	1	0	0	0
$t(u(t-1)-u(t-2))$	0	t	0	0
$(4-t)(u(t-2)-u(t-3))$	0	0	$4-t$	0
$u(t-3)$	0	0	0	1

$$f(t) = [1-u(t-1)] + t[u(t-1)-u(t-2)] + (4-t)[u(t-2)-u(t-3)] + u(t-3)$$

$$= 1 + (t-1)\,u(t-1) + (4-2t)\,u(t-2) + (t-3)\,u(t-3)$$

(6) $f(t) = t^2\left(u(t-1) - u(t-2)\right)$

(a) $= \begin{cases} t^2 & \text{if } 1<t<2 \\ 0 & \text{otherwise} \end{cases}$



(b) $\mathcal{L}\{f(t)\} = \mathcal{L}\{u(t-1)\,t^2\} - \mathcal{L}\{t^2 u(t-2)\}$

$= e^{-s}\mathcal{L}\{(t+1)^2\} - e^{-2s}\mathcal{L}\{(t+2)^2\}$

$= e^{-s}\mathcal{L}\{1+2t+t^2\} - e^{-2s}\mathcal{L}\{4+4t+t^2\}$

$= e^{-s}\left(\frac{1}{s} + \frac{2}{s^2} + \frac{2}{s^3}\right) - e^{-2s}\left(\frac{4}{s} + \frac{4}{s^2} + \frac{2}{s^3}\right)$

for $s>0$.

(7)

$$F(s) = \frac{(2s^2+4s+3)}{(s+1)^2(s+2)} = \frac{A}{s+1} + \frac{B}{(s+1)^2} + \frac{C}{(s+2)}$$

(partial fractions).

$$(2s^2+4s+3) = A(s+1)(s+2) + B(s+2) + C(s+1)^2.$$

$s=-1$: $\quad 2-4+3 = B \quad$ & $B=1$.

$s=-2$: $\quad 8-8+3 = C \quad$ & $C=3$.

Coeff of s^2: $\quad 2 = A + C$, $A = 2 - C = 2-3 = -1$.

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{-1}{s+1} + \frac{1}{(s+1)^2} + \frac{3}{s+2}\right\}$$

$$= -e^{-t} + te^{-t} + 3e^{-2t}$$

$$= (t-1)e^{-t} + 3e^{-2t}.$$

Using the formula

$$\mathcal{L}\{u(t-t_0)f(t-t_0)\} = e^{-st_0}\mathcal{L}\{f(t)\} = e^{-st_0}F(s)$$

with $t_0 = 3$ we have

$$\mathcal{L}^{-1}\left\{e^{-3s}\frac{(2s^2+4s+3)}{(s+1)^2(s+2)}\right\} = u(t-3)f(t-3)$$

$$= u(t-3)\left[(t-4)e^{-(t-3)} + 3e^{-2(t-3)}\right].$$